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On Weakly e^*-θ-open Functions and Their Characterizations

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Abstract. The main aim of this paper is to introduce and study a new class of weakly open functions called weakly e^*-θ-open functions via e^*-θ-open sets [13], which is stronger than the concept of weakly e^*-open functions [22]. Moreover, we obtain various characterizations of weakly e^*-θ-open functions and investigate some of their fundamental properties. In additon, we investigate not only the relationships of these functions with some other types of existing topological functions, but also several basic results related to connectedness, $e^*\theta$-connectedness [5] and hyperconnectedness [18].

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1. Introduction and Preliminaries

Open sets are, as well known, one of the most important notions of pure mathematics. The concept of open function defined through open sets is also the subject of topology, which has permeated all branches of science. In the process of time, mathematicians have found it very useful to generalize open functions. First in 1984, D.A. Rose [23] introduced weakly open functions, which is a more general concept than the open functions and obtained some results regarding this concept. In recent years, many authors have studied on generalizations of weakly open functions such as weakly preopen functions [6], weakly semiopen functions [7], weakly b-open functions [20], weakly b-θ-open functions [19] (weakly BR-open functions [8]), weakly eR-open functions [21] and weakly e^*-open functions [22].

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open, clopen) sets of X is denoted by $C(X)$ (resp. $O(X), CO(X)$). A subset A is said to be regular open [26] (resp. regular closed [26]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point [27] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure [27] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [27] and the complement of a δ-closed set is called δ-open [27]. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called semiopen [16] (resp. preopen [17], b-open [1], e-open [10], e^*-open [12], a-open [9]) if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subseteq cl(int(cl_\delta(A)))$, $A \subseteq int(cl(int_\delta(A))))$. The complement of a semiopen (resp. preopen, b-open, e-open, e^*-open, a-open) set is called semiclosed [16] (resp. preclosed [17], b-closed [1], e-closed [10], e^*-closed [12], a-closed [9]). The intersection of all semiclosed (resp. preclosed, b-closed, e-closed, e^*-closed, a-closed) sets of X containing A is called the semi-closure [16] (resp. pre-closure [17], b-closure [1], e-closure [10], e^*-closure [12], a-closure [9]) of A and is denoted

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by $scl(A)$ (resp. $pcl(A)$, $bcl(A)$, $e\text{-}cl(A)$, $e^*\text{-}cl(A)$, $a\text{-}cl(A)$). The union of all semiopen (resp. preopen, b-open, e-open, e^*-open, a-open) sets of X contained in A is called the semi-interior [16] (resp. pre-interior [17], b-interior [1], e-interior [10], e^*-interior [12], a-interior [9]) of A and is denoted by $sint(A)$ (resp. $pint(A)$, $bint(A)$, $e\text{-}int(A)$, $e^*\text{-}int(A)$, $a\text{-}int(A)$).

A point x of X is called a θ-cluster (e^*-θ-cluster) point of A if $cl(U) \cap A \neq \emptyset$ ($e^*\text{-}cl(U) \cap A \neq \emptyset$) for every open ($e^*$-open) set U containing x. The set of all θ-cluster (e^*-θ-cluster) points of A is called the θ-closure [27] (e^*-θ-closure [13]) of A and is denoted by $cl_\theta(A)$ ($e^*\text{-}cl_\theta(A)$). A subset A is said to be θ-closed (e^*-θ-closed) if $A = cl_\theta(A)$ ($A = e^*\text{-}cl_\theta(A)$). The complement of a θ-closed (e^*-θ-closed) set is called a θ-open [27] (e^*-θ-open [13]). A point x of X said to be a θ-interior [27] (e^*-θ-interior [13]) point of a subset A, denoted by $int_\theta(A)$ ($e^*\text{-}int_\theta(A)$), if there exists an open (e^*-open) set U of X containing x such that $cl(U) \subseteq A$ ($e^*\text{-}cl(U) \subseteq A$).

A subset A is said to be e^*-regular [13] set if it is e^*-open and e^*-closed. Also it is noted in [13] that

$$e^*\text{-regular} \Rightarrow e^*\text{-}\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all e^*-θ-open (resp. e^*-θ-closed, e^*-open, e^*-closed, e^*-regular, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, b-open, b-closed, e-open, e-closed, a-open, a-closed) subsets of X is denoted by $e^*\theta O(X)$ (resp. $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $e^*R(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $PC(X)$, $BO(X)$, $BC(X)$, $eO(X)$, $eC(X)$, $aO(X)$, $aC(X)$). The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, e^*-regular, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, b-open, b-closed, e-open, e-closed, a-open, a-closed) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $e^*\theta O(X,x)$, $e^*\theta C(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $e^*R(X,x)$, $RO(X,x)$, $RC(X,x)$, $\delta O(X,x)$, $\delta C(X,x)$, $\theta O(X,x)$, $\theta C(X,x)$, $SO(X,x)$, $SC(X,x)$, $PO(X,x)$, $PC(X,x)$, $BO(X,x)$, $BC(X,x)$, $eO(X,x)$, $eC(X,x)$, $aO(X,x)$, $aC(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article. The following basic properties of e^*-θ-closure is useful in the sequel:

Lemma 1.1 ([13, 14]). *Let A be a subset of a space X, then the following properties hold:*
(1) $A \subseteq e^*\text{-}cl(A) \subseteq e^*\text{-}cl_\theta(A)$,
(2) $e^*\text{-}cl_\theta(A)$ is e^*-θ-closed,
(3) $x \in e^*\text{-}cl_\theta(A)$ if and only if $A \cap U \neq \emptyset$ for every $e^*R(X,x)$,
(4) $e^*\text{-}cl_\theta(A) = \cap\{V | (A \subseteq V)(V \subseteq e^*R(X))\}$,
(5) If $A \subseteq B$, then $e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(B)$,
(6) $e^*\text{-}cl_\theta(e^*\text{-}cl_\theta(A)) = e^*\text{-}cl_\theta(A)$,
(7) $e^*\text{-}cl_\theta(X \setminus A) = X \setminus e^*\text{-}int_\theta(A)$,
(8) Any intersections of e^*-θ-closed sets is e^*-θ-closed and any union of e^*-θ-open sets is e^*-θ-open,
(9) A is e^*-θ-open in X if and only if for each $x \in A$ there exists an e^*-regular set U containing x such that $x \in U \subseteq A$,
(10) If A is e^*-θ-open, then $A = e^*\text{-}int_\theta(A)$.

Lemma 1.2 ([13]). *Let X be a topological space and $\mathcal{A} \subseteq 2^X$, then the following properties hold:*
(a) $\cup\{e^*\text{-}int_\theta(A) | A \in \mathcal{A}\} \subseteq e^*\text{-}int_\theta(\cup\{A | A \in \mathcal{A}\})$,
(b) $e^*\text{-}cl_\theta(\cap\{A | A \in \mathcal{A}\}) \subseteq \cap\{e^*\text{-}cl_\theta(A) | A \in \mathcal{A}\}$.

Lemma 1.3 ([27]). *Let X be a topological space and $A \subseteq X$, then the following properties hold:*
(a) If A is open, then $cl(A) = cl_\theta(A)$,
(b) If A is closed, then $int(A) = int_\theta(A)$.

Definition 1.4. A function $f : X \to Y$ is called:
(a) e^*-θ-open [3] if $f[U]$ is e^*-θ-open in Y for each open set U of X,
(b) contra e^*-θ-open if $f[U]$ is e^*-θ-closed in Y for each open set U of X,
(c) contra e^*-θ-closed if $f[U]$ is e^*-θ-open in Y for each closed set U of X,
(d) strongly continuous ([2, 15]) if for every subset A of X, $f[cl(A)] \subseteq f[A]$,
(e) almost open (in the sense of Singal [25]) if the image of each regular open subset U of X is open set in Y .

2. Weakly e^*-θ-open Functions

In this section, we define the concept of weakly e^*-θ-open and investigate some basic properties of them.

Definition 2.1. A function $f : X \to Y$ is said to be weakly e^*-θ-open if $f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)])$ for each open set U of X.

Definition 2.2. A function $f : X \to Y$ is called:
(a) weakly open [23] if $f[U] \subseteq int(f[cl(U)])$ for each open set U of X,
(b) weakly preopen [6] if $f[U] \subseteq pint(f[cl(U)])$ for each open set U of X,
(c) weakly semiopen [7] if $f[U] \subseteq sint(f[cl(U)])$ for each open set U of X,
(d) weakly b-open [20] if $f[U] \subseteq bint(f[cl(U)])$ for each open set U of X,
(e) weakly b-θ-open [19] (weakly BR-open [8]) if $f[U] \subseteq bint_\theta(f[cl(U)])$ for each open set U of X,
(f) weakly eR-open [21] if $f[U] \subseteq e\text{-}int_\theta(f[cl(U)])$ for each open set U of X,
(g) weakly e^*-open [22] if $f[U] \subseteq e^*\text{-}int(f[cl(U)])$ for each open set U of X.

Remark 2.3. From Lemma 1.1, Definition 2.1 and Definition 2.2, we have the following diagram. None of these implications is reversible as shown by the following example. Examples regarding other implications can be found related articles.

$$
\begin{array}{ccccccc}
 & & & & \text{weakly } eR\text{-open} & & e^*\text{-}\theta\text{-open} \\
 & & & & \downarrow & \searrow & \downarrow \\
 & & \text{weakly preopen} & \to & \text{weakly } e\text{-open} & & \text{weakly } e^*\text{-}\theta\text{-open} \\
 & \nearrow & & \searrow & & \searrow & \downarrow \\
\text{weakly open} & & & & \text{weakly } b\text{-open} & & \text{weakly } e^*\text{-open} \\
 & \searrow & & \nearrow & \uparrow & & \\
 & & \text{weakly semiopen} & & \text{weakly } b\text{-}\theta\text{-open} & &
\end{array}
$$

Example 2.4. Let $X = \{a, b, c, d\}$. Define two topologies $\tau = \{\emptyset, X, \{a, b\}\}$ and $\tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$ on X. Then the identity function $f : (X, \tau) \to (X, \tau^*)$ is weakly e^*-θ-open which is not e^*-θ-open.

Example 2.5. Let $X = \{a, b, c, d\}$. Define two topologies $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and $\tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$ on X. Then the identity function $f : (X, \tau) \to (X, \tau^*)$ is weakly e^*-θ-open which is not weakly e-θ-open.

Example 2.6. Let $X = \{1, 2, 3, 4, 5\}$ and $\tau = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}, \{3, 4\}, \{1, 3, 4\}\}$ on X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is weakly e^*-open which is not weakly e^*-θ-open.

Theorem 2.7. *Let $f : X \to Y$ be a function, then the following statements are equivalent:*
(a) f is weakly e^*-θ-open,
(b) $f[int_\theta(A)] \subseteq e^*\text{-}int_\theta(f[A])$ for every subset A of X,
(c) $int_\theta(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}int_\theta(B)]$ for every subset B of Y,
(d) $f^{-1}[e^*\text{-}cl_\theta(B)] \subseteq cl_\theta(f^{-1}[B])$ for every subset B of Y,
(e) $f[int(F)] \subseteq e^*\text{-}int_\theta(f[F])$ for each closed subset F of X,
(f) $f[int(cl(U))] \subseteq e^*\text{-}int_\theta(f[cl(U)])$ for each open subset U of X,
(g) $f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)])$ for every regular open subset U of X,
(h) $f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)])$ for every α-open subset U of X,
(i) For each $x \in X$ and each open set U of X containing x, there exists an e^*-θ-open set V of Y containing $f(x)$ such that $V \subseteq f[cl(U)]$.

Proof. $(a) \Rightarrow (b)$: Let $A \subseteq X$ and $y \in f[int_\theta(A)]$.

$$y \in f[int_\theta(A)] \Rightarrow (\exists x \in int_\theta(A))(y = f(x)) \Rightarrow (\exists U \in O(X, x))(U \subseteq cl(U) \subseteq A)$$

$$\left.\begin{array}{r} \Rightarrow (U \in O(X, x))(y = f(x) \in f[U] \subseteq f[cl(U)] \subseteq f[A]) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow y = f(x) \in f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)]) \subseteq e^*\text{-}int_\theta(f[A]).$$

$(b) \Rightarrow (c)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f\left[int_\theta(f^{-1}[B])\right] \subseteq e^*\text{-}int_\theta(f[f^{-1}[B]]) \subseteq e^*\text{-}int_\theta(B) \Rightarrow int_\theta(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}int_\theta(B)].$$

$(c) \Rightarrow (d)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow Y \setminus B \subseteq Y \\ \text{Hypothesis} \end{array}\right\} \Rightarrow int_\theta(X \setminus f^{-1}[B]) = int_\theta(f^{-1}[Y \setminus B]) \subseteq f^{-1}[e^*\text{-}int_\theta(Y \setminus B)] = f^{-1}[Y \setminus e^*\text{-}cl_\theta(B)]$$

$\Rightarrow X \setminus cl_\theta(f^{-1}[B]) \subseteq X \setminus f^{-1}[e^*\text{-}cl_\theta(B)]$

$\Rightarrow f^{-1}[e^*\text{-}cl_\theta(B)] \subseteq cl_\theta(f^{-1}[B])$.

$(d) \Rightarrow (e)$: Let $F \in C(X)$.

$$\left.\begin{array}{r} F \in C(X) \Rightarrow X \setminus F \in O(X) \Rightarrow cl_\theta(f^{-1}f[F]) = cl_\theta(X \setminus f^{-1}[f[F]]) \subseteq cl_\theta(X \setminus F) = cl(X \setminus F)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\begin{array}{rcl} \Rightarrow X \setminus f^{-1}[e^*\text{-}int_\theta(f[F])] & = & f^{-1}[e^*\text{-}cl_\theta(Y \setminus f[F])] \subseteq cl_\theta(f^{-1}[Y \setminus f[F]]) \\ & = & X \setminus int_\theta(f^{-1}(f[F])) \subseteq X \setminus int_\theta(F) = X \setminus int(F) \end{array}$$

$\Rightarrow int(F) \subseteq f^{-1}[e^*\text{-}int_\theta(f[F])]$

$\Rightarrow f\,[int(F)] \subseteq e^*\text{-}int_\theta(f[F])$.

$(e) \Rightarrow (f)$ and $(f) \Rightarrow (g)$: Straightforward.

$(g) \Rightarrow (h)$: Let $U \in \alpha O(X)$.

$$\left.\begin{array}{r} U \in \alpha O(X) \Rightarrow U \subseteq int(cl(int(U))) \in RO(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow f[U] \subseteq f[int(cl(int(U)))] \subseteq e^*\text{-}int_\theta(f[cl(int(cl(int(U))))]) = e^*\text{-}int_\theta(f[cl(int(U))]) \subseteq e^*\text{-}int_\theta(f[cl(U)])$.

$(h) \Rightarrow (i)$: $x \in X$ and $U \in O(X)$.

$$\left.\begin{array}{r} (x \in X)(U \in O(X)) \Rightarrow U \in \alpha O(X, x) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (V := e^*\text{-}int_\theta(f[cl(U)]) \in e^*\theta O(Y, f(x)))(f[U] \subseteq V \subseteq f[cl(U)]).$$

$(i) \Rightarrow (a)$: Let $U \in O(X)$ and $y \in f[U]$.

$$\left.\begin{array}{r} (U \in O(X))(y \in f[U]) \Rightarrow (\exists x \in U \in O(X))(y = f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists V \in e^*\theta O(Y, y))(V \subseteq f[cl(U)])$$

$\Rightarrow (\exists V \in e^*\theta O(Y, y))(V = e^*\text{-}int_\theta(V) \subseteq e^*\text{-}int_\theta(f[cl(U)]))$

$\Rightarrow y \in e^*\text{-}int_\theta(f[cl(U)]))$. □

Theorem 2.8. *Let $f : X \to Y$ be a bijective function, then the following statements are equivalent:*
(a) f is weakly e^*-θ-open,
(b) For each $x \in X$ and each open set U of X containing x, there exists an e^*-regular set V containing $f(x)$ such that $V \subseteq f[cl(U)]$,
(c) e^*-$cl_\theta(f\,[int(cl(U))]) \subseteq f\,[cl(U)]$ for each subset U of X,
(d) e^*-$cl_\theta(f\,[int(F)]) \subseteq f\,[F]$ for each regular closed subset F of X,
(e) e^*-$cl_\theta(f\,[U]) \subseteq f\,[cl(U)]$ for each open subset U of X,
(f) e^*-$cl_\theta(f\,[U]) \subseteq f\,[cl(U)]$ for each preopen subset U of X,
(g) $f\,[U] \subseteq e^*$-$int_\theta(f\,[cl(U)])$ for each preopen subset U of X,
(h) $f^{-1}\,[e^*\text{-}cl_\theta(B)] \subseteq cl_\theta(f^{-1}\,[B])$ for each subset B of Y,
(i) e^*-$cl_\theta(f\,[U]) \subseteq f\,[cl_\theta(U)]$ for each subset U of X,
(j) e^*-$cl_\theta(f\,[int(cl_\theta(U))]) \subseteq f\,[cl_\theta(U)]$ for each subset U of X.

Proof. $(a) \Rightarrow (b)$: Let $U \in O(X, x)$.

$$\left.\begin{array}{r} \left.\begin{array}{r} U \in O(X, x) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f(x) \in f\,[U] \subseteq e^*\text{-}int_\theta(f\,[cl(U)]) \in e^*\theta O(Y, f(x)) \\ \text{Lemma 1.1(9)} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in e^*R(Y, f(x)))(V \subseteq e^*\text{-}int_\theta(f\,[cl(U)]) \subseteq f\,[cl(U)])$.

$(b) \Rightarrow (c)$: Let $U \subseteq X$ and $y \notin f[cl(U)]$.

$$\left.\begin{array}{r} y \notin f\,[cl(U)] \Rightarrow y \in Y \setminus f[cl(U)] = f[X \setminus cl(U)] \\ f \text{ is bijective} \end{array}\right\} \Rightarrow (\exists! x \in X \setminus cl(U))(y = f(x))$$

$\Rightarrow (\exists G \in O(X,x))(G \cap U = \emptyset) \Rightarrow (\exists G \in O(X,x))(cl(G) \cap int(cl(U)) = \emptyset)$

$$\Rightarrow \left.\begin{array}{r}(\exists G \in O(X,x))(f[cl(G) \cap int(cl(U))] = \emptyset)\\ f \text{ is bijective}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists G \in O(X,x))(f[cl(G)] \cap f[int(cl(U))] = \emptyset)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in e^*R(Y, y = f(x)))(V \cap f[int(cl(U))] = \emptyset)$
$\Rightarrow y \notin e^*\text{-}cl_\theta(f[int(cl(U))])$.

$(c) \Rightarrow (d)$: Let $F \in RC(X)$.

$$\left.\begin{array}{r}F \in RC(X) \Rightarrow e^*\text{-}cl_\theta(f[int(F)]) = e^*\text{-}cl_\theta(f[int(cl(int(F)))])\\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[int(F)]) \subseteq f[cl(int(F))] = f[F].$$

$(d) \Rightarrow (e)$: Let $U \in O(X)$.

$$\left.\begin{array}{r}(U \in O(X))(cl(U) \in RC(X))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[U]) \subseteq e^*\text{-}cl_\theta(f[int(cl(U))]) \subseteq f[cl(U)].$$

$(e) \Rightarrow (f)$: Let $U \in PO(X)$.

$$\left.\begin{array}{r}U \in PO(X) \Rightarrow (U \subseteq int(cl(U)))(int(cl(U)) \in O(X))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$\Rightarrow e^*\text{-}cl_\theta(f[U]) \subseteq e^*\text{-}cl_\theta(f[int(cl(U))]) \subseteq f[cl(int(cl(U)))] \subseteq f[cl(U)]$.

$(f) \Rightarrow (g)$: Let $U \in PO(X)$.

$$\left.\begin{array}{r}U \in PO(X) \Rightarrow X \setminus cl(U) \in O(X)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow \left.\begin{array}{r}e^*\text{-}cl_\theta(f[X \setminus cl(U)]) \subseteq f[cl(X \setminus cl(U))]\\ f \text{ is bijective}\end{array}\right\} \Rightarrow$$

$\Rightarrow Y \setminus e^*\text{-}int_\theta(f[cl(U)]) = e^*\text{-}cl_\theta(Y \setminus f[cl(U)]) \subseteq f[X \setminus int(cl(U))] = Y \setminus f[int(cl(U))]$
$\Rightarrow f[U] \subseteq f[int(cl(U))] \subseteq e^*\text{-}int_\theta(f[cl(U)])$.

$(g) \Rightarrow (h)$: Let $x \notin cl_\theta(f^{-1}[B])$.
$x \notin cl_\theta(f^{-1}[B]) \Rightarrow (\exists U \in O(X,x))(cl(U) \cap f^{-1}[B] = \emptyset) \Rightarrow (\exists U \in O(X,x))(f[cl(U) \cap f^{-1}[B]] = \emptyset)$

$$\overset{f \text{ is bijective}}{\Rightarrow} \left.\begin{array}{r}(\exists U \in O(X,x))(f[cl(U)] \cap B = \emptyset) \Rightarrow (\exists U \in O(X,x))(e^*\text{-}int_\theta(f[cl(U)]) \cap e^*\text{-}int_\theta(B) = \emptyset)\\ U \in O(X,x) \Rightarrow U \in PO(X)\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in O(X,x))(f[U] \cap e^*\text{-}int_\theta(B) = \emptyset)$
$\Rightarrow (\exists U \in O(X,x))(U \cap f^{-1}[e^*\text{-}int_\theta(B)] = \emptyset)$
$\Rightarrow x \notin f^{-1}[e^*\text{-}int_\theta(B)]$.

$(h) \Rightarrow (i)$: Let $U \subseteq X$.

$$\left.\begin{array}{r}U \subseteq X \Rightarrow f[U] \subseteq Y\\ \text{Hypothesis}\end{array}\right\} \Rightarrow f^{-1}[e^*\text{-}cl_\theta(f[U])] \subseteq cl_\theta(f^{-1}[f[U]]) \overset{f \text{ is bijective}}{=} cl_\theta(U) \Rightarrow e^*\text{-}cl_\theta(f[U]) \subseteq f[cl_\theta(U)].$$

$(i) \Rightarrow (j)$: Let $U \subseteq X$.

$$\left.\begin{array}{r}U \subseteq X \Rightarrow int(cl_\theta(U)) \subseteq X\\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[int(cl_\theta(U))]) \subseteq f[cl_\theta(int(cl_\theta(U)))] \overset{\text{Lemma 1.3(a)}}{=} f[cl(int(cl_\theta(U)))] \subseteq f[cl_\theta(U)].$$

$(j) \Rightarrow (a)$: Let $U \in O(X)$.

$$\left.\begin{array}{r}U \in O(X) \Rightarrow \setminus cl(U) \in O(X)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[int(cl_\theta(\setminus cl(U)))]) \subseteq f[cl_\theta(\setminus cl(U))]$$

$\Rightarrow \setminus e^*\text{-}int_\theta(f[cl(int_\theta(cl(U)))]) \subseteq \setminus f[int_\theta(cl(U))]$
$\Rightarrow f[U] \subseteq f[int_\theta(cl(U))] \subseteq e^*\text{-}int_\theta(f[cl(int_\theta(cl(U)))]) \subseteq e^*\text{-}int_\theta(f[cl(U)])$. □

Theorem 2.9. *If X is a regular space and $f : X \to Y$ is a bijective function, then the following statements are equivalent:*
(a) f is weakly e^*-θ-open,
(b) For each θ-open set A in X, $f[A]$ is e^*-θ-open in Y,

(c) For any set B of Y and any θ-closed set A in X containing $f^{-1}[B]$, there exists an e^*-θ-closed set F in Y containing B such that $f^{-1}[F] \subseteq A$.

Proof. $(a) \Rightarrow (b)$: Let $A \in \theta O(X)$.

$$\left.\begin{array}{r} A \in \theta O(X) \Rightarrow Y \setminus f[A] \subseteq Y \\ \text{Theorem 2.8(h)} \end{array}\right\} \Rightarrow f^{-1}[e^*\text{-}cl_\theta(Y \setminus f[A])] \subseteq cl_\theta(f^{-1}[Y \setminus f[A]])$$

$\Rightarrow X \setminus f^{-1}[e^*\text{-}int_\theta(f[A])] \subseteq cl_\theta(X \setminus A) = X \setminus A$
$\Rightarrow A \subseteq f^{-1}[e^*\text{-}int_\theta(f[A])]$
$\Rightarrow f[A] \subseteq e^*\text{-}int_\theta(f[A])$.

$(b) \Rightarrow (c)$: Let $B \subseteq Y$ and $A \in \theta C(X)$ such that $f^{-1}[B] \subseteq A$.

$$\left.\begin{array}{r} (B \subseteq Y)(A \in \theta C(X))(f^{-1}[B] \subseteq A) \overset{f \text{ is bijective}}{\Rightarrow} (X \setminus A \in \theta O(X))(B \subseteq Y \setminus f[X \setminus A]) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{r} (f[X \setminus A] \in e^*\theta O(Y))(B \subseteq Y \setminus f[X \setminus A]) \\ F := Y \setminus f[X \setminus A] \end{array}\right\} \Rightarrow (F \in e^*\theta C(X))(B \subseteq F)(f^{-1}[F] = f^{-1}[Y \setminus f[X \setminus A]] = A).$$

$(c) \Rightarrow (a)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} \left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ X \text{ is regular} \end{array}\right\} \Rightarrow f^{-1}[B] \subseteq cl_\theta(f^{-1}[B]) \in \theta C(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists F \in e^*\theta C(Y))(B \subseteq F)(f^{-1}[e^*\text{-}cl_\theta(B)] \subseteq f^{-1}[F] \subseteq cl_\theta(f^{-1}[B]))$

Since f is bijective, then by Theorem 2.8(h) f is weakly e^*-θ-open. □

Theorem 2.10. *If X is a regular space and $f : X \to Y$ is a bijective function, then the following statements are equivalent:*
(a) f is weakly e^*-θ-open,
(b) f is e^*-θ-open,
(c) For each $x \in X$ and each open set U of X containing x, there exists an e^*-open set V of Y containing $f(x)$ such that e^*-$cl_\theta(V) \subseteq f[U]$.

Proof. $(a) \Rightarrow (b)$: Let $U \in O(X)$.

$$\left.\begin{array}{r} (U \in O(X))(y \in f[U]) \Rightarrow (\exists U \in O(X,x))(f(x) = y) \\ X \text{ is regular} \end{array}\right\} \Rightarrow (U \in \theta O(X,x))(f(x) = y) \Rightarrow (x \in int_\theta(U))(f(x) = y)$$

$$\Rightarrow \left.\begin{array}{r} (\exists W \in O(X,x))(cl(W) \subseteq U)(f(x) = y) \Rightarrow (W \in O(X,x))(f[cl(W)] \subseteq f[U])(f(x) = y) \\ f \text{ is weakly } e^*\text{-}\theta\text{-open} \end{array}\right\} \overset{\text{Theorem 2.8(b)}}{\Rightarrow}$$

$\Rightarrow (\exists V \in e^*R(Y,y))(V \subseteq f[cl(W)] \subseteq f[U]) \overset{\text{Lemma 1.1(9)}}{\Rightarrow} y \in e^*\text{-}int_\theta(f[U])$.

$(b) \Rightarrow (c)$: Let $x \in X$ and $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X,x) \\ f \text{ is } e^*\text{-}\theta\text{-open} \end{array}\right\} \Rightarrow f[U] \in e^*\theta O(Y, f(x)) \overset{\text{Lemma 1.1(9)}}{\Rightarrow} (\forall y := f(x) \in f[U])(\exists V \in e^*R(Y,y))(V \subseteq f[U])$$

$\Rightarrow (V \in e^*O(Y,y))(e^*\text{-}cl_\theta(V) = V \subseteq f[U])$.

$(c) \Rightarrow (a)$: Let $U \in O(X)$.

$$\left.\begin{array}{r} y \in f[U] \Rightarrow (\exists! x \in U)(f(x) = y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists V \in e^*O(Y, f(x)))(e^*\text{-}cl(V) \subseteq e^*\text{-}cl_\theta(V) \subseteq f[U] \subseteq f[cl(U)])$$

$\Rightarrow y \in e^*\text{-}int_\theta(f[cl(U)])$. □

Theorem 2.11. *Let $f : X \to Y$ be a bijective weakly e^*-θ-open function, then the following properties hold:*
(a) If F is θ-closed in X, then $f[F]$ is e^*-θ-closed in Y,
(b) If F is θ-open in X, then $f[F]$ is e^*-θ-open in Y.

Proof. (a) Let $F \in \theta C(X)$.

$$\left.\begin{array}{r} F \in \theta C(X) \Rightarrow F = cl_\theta(F) \\ f \text{ is bijective weakly } e^*\text{-}\theta\text{-open} \end{array}\right\} \overset{\text{Theorem 2.8(h)}}{\Rightarrow} \left.\begin{array}{r} e^*\text{-}cl_\theta(f[F]) \subseteq f[cl_\theta(F)] = f[F] \\ f[F] \subseteq e^*\text{-}cl_\theta(f[F]) \end{array}\right\} \Rightarrow f[F] = e^*\text{-}cl_\theta(f[F])$$

$\Rightarrow f[F] \in e^*\theta C(Y)$.

(b) It can be proved similarly. □

Definition 2.12. A function $f : X \to Y$ is called weakly e^*-θ-closed if for each closed set F of X such that e^*-$cl_\theta(f[int(F)]) \subseteq f[F]$.

Theorem 2.13. *Let $f : X \to Y$ be a bijective function, then the following statements are equivalent:*
(a) f is weakly e^*-θ-closed,
(b) e^*-$cl_\theta(f[U]) \subseteq f[cl(U)]$ for each open subset U of X,
(c) e^*-$cl_\theta(f[U]) \subseteq f[cl(U)]$ for each regular open subset U of X,
(d) For each subset F in Y and each open subset U in X with $f^{-1}[F] \subseteq U$, there exists an e^*-θ-open set A in Y with $F \subseteq A$ and $f^{-1}[A] \subseteq cl(U)$,
(e) For each $y \in Y$ and each open subset U in X with $f^{-1}(y) \subseteq U$, there exists an e^*-θ-open set A in Y containing y and $f^{-1}[A] \subseteq cl(U)$,
(f) e^*-$cl_\theta(f[int(cl(U))]) \subseteq f[cl(U)]$ for each open set U of X,
(g) e^*-$cl_\theta(f[int(cl_\theta(U))]) \subseteq f[cl_\theta(U)]$ for each open set U of X.

Proof. $(a) \Rightarrow (b)$: Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \Rightarrow e^*\text{-}cl_\theta(f[U]) = e^*\text{-}cl_\theta(f[int(U)]) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[U]) \subseteq e^*\text{-}cl_\theta(f[int(cl(U))]) \subseteq f[cl(U)].$$

$(b) \Rightarrow (c)$: Straightforward.

$(c) \Rightarrow (d)$: Let $F \subseteq Y$ and $U \in O(X)$ such that $f^{-1}[F] \subseteq U$.

$$\left.\begin{array}{r} f^{-1}[F] \subseteq U \subseteq cl(U) \Rightarrow f^{-1}[F] \cap cl(X \setminus cl(U)) = \varnothing \\ f \text{ is bijective} \end{array}\right\} \Rightarrow \left.\begin{array}{r} F \cap f[cl(X \setminus cl(U))] = \varnothing \\ U \in O(X) \Rightarrow X \setminus cl(U) \in RO(X) \end{array}\right\} \overset{\text{Hypothesis}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow F \cap e^*\text{-}cl_\theta(f[X \setminus cl(U)]) = \varnothing \Rightarrow F \subseteq Y \setminus e^*\text{-}cl_\theta(f[X \setminus cl(U)]) \\ A := Y \setminus e^*\text{-}cl_\theta(f[X \setminus cl(U)]) \end{array}\right\} \Rightarrow$$

$\Rightarrow (A \in e^*\theta O(Y))(F \subseteq A)(f^{-1}[A] = f^{-1}[Y \setminus e^*\text{-}cl_\theta(f[X \setminus cl(U)])] \subseteq X \setminus f^{-1}[f[X \setminus cl(U)]] = cl(U))$.

$(d) \Rightarrow (e)$: Let $y \in Y$ and $U \in O(X)$ such that $f^{-1}(y) \subseteq U$.

$$\left.\begin{array}{r} f^{-1}(y) \subseteq U \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists A \in e^*\theta O(Y))(\{y\} \subseteq A)(f^{-1}[A] \subseteq cl(U)) \Rightarrow (\exists A \in e^*\theta O(Y, y))(f^{-1}[A] \subseteq cl(U)).$$

$(e) \Rightarrow (f)$: Let $U \in O(X)$ and $y \notin f[cl(U)]$.

$$\left.\begin{array}{r} \left.\begin{array}{r} y \notin f[cl(U)] \Rightarrow y \in Y \setminus f[cl(U)] \Rightarrow f^{-1}(y) \subseteq X \setminus cl(U) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists A \in e^*\theta O(Y, y))(f^{-1}[A] \subseteq cl(X \setminus cl(U))) \\ f \text{ is bijective} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists A \in e^*\theta O(Y, y))(A \subseteq f[cl(X \setminus cl(U))] = Y \setminus f[int(cl(U))])$
$\Rightarrow (\exists A \in e^*\theta O(Y, y))(A \cap f[int(cl(U))] = \varnothing)$
$\Rightarrow y \notin e^*\text{-}cl_\theta(f[int(cl(U))])$.

$(f) \Rightarrow (g)$: Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \overset{\text{Lemma 1.3}(a)}{\Rightarrow} cl_\theta(U) = cl(U) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[int(cl_\theta(U))]) \subseteq f[cl_\theta(U)].$$

$(g) \Rightarrow (a)$: Let $F \in C(X)$.

$$\left.\begin{array}{r} F \in C(X) \Rightarrow int(F) \in O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[int(cl_\theta(int(F)))]) \subseteq f[cl_\theta(int(F))]$$

$$\Rightarrow e^*\text{-}cl_\theta(f[int(F)]) \subseteq e^*\text{-}cl_\theta(f[int(cl_\theta(int(F)))]) \subseteq f[cl_\theta(int(F))] = f[cl(int(F))] \subseteq f[F].$$ □

Theorem 2.14. *If $f : X \to Y$ is weakly e^*-θ-open and strongly continuous, then f is e^*-θ-open.*

Proof. Let $U \in O(X)$.

$$\left.\begin{array}{r} \left.\begin{array}{r} U \in O(X) \\ f \text{ is weakly } e^*\text{-}\theta\text{-open} \end{array}\right\} \Rightarrow f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)]) \\ f \text{ is strongly continuous} \end{array}\right\} \Rightarrow f[U] \subseteq e^*\text{-}int_\theta(f[cl(U)]) \subseteq e^*\text{-}int_\theta(f[U]).$$ □

Remark 2.15. The following example shows that strong continuity is not decomposition of e^*-θ-openness. Namely, an e^*-θ-open function need not be strongly continuous.

Example 2.16. Let $X = \{a, b\}$. Let τ be the indiscrete topology and let τ^* be the discrete topology on X. Then, the identity function $f : (X, \tau) \to (X, \tau^*)$ is an e^*-θ-open which is not strongly continuous.

Theorem 2.17. *If $f : X \to Y$ is an almost open function, then f is weakly e^*-θ-open.*

Proof. Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \Rightarrow U \subseteq int(cl(U)) \in RO(X) \\ f \text{ is almost open} \end{array}\right\} \Rightarrow f[U] \subseteq f[int(cl(U))] \in O(Y)$$

$$\Rightarrow f[U] \subseteq f[int(cl(U))] = int(f[int(cl(U))])$$

$$\Rightarrow f[U] \subseteq f[int(cl(U))] \subseteq int(f[cl(U)]) \subseteq e^*\text{-}int_\theta(f[cl(U)]).$$ □

Remark 2.18. The converse of above theorem is not true in general. Namely, a weakly e^*-θ-open function need not be almost open function.

Example 2.19. Consider the same topologies on X in Example 2.5. Then the identity function $f : (X, \tau) \to (X, \tau^*)$ is weakly e^*-θ-open which is not almost open.

Theorem 2.20. *If $f : X \to Y$ is contra e^*-θ-closed, then f is a weakly e^*-θ-open function.*

Proof. Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \Rightarrow cl(U) \in C(X) \\ f \text{ is contra } e^*\text{-}\theta\text{-closed} \end{array}\right\} \Rightarrow f[cl(U)] \in e^*\theta O(Y) \Rightarrow f[U] \subseteq f[cl(U)] = e^*\text{-}int_\theta(f[cl(U)]).$$ □

Theorem 2.21. *If $f : X \to Y$ is bijective contra e^*-θ-open, then f is a weakly e^*-θ-open function.*

Proof. Let $U \in O(X)$.

$$\left.\begin{array}{r} U \in O(X) \\ f \text{ is contra } e^*\text{-}\theta\text{-open} \end{array}\right\} \Rightarrow f[U] \in e^*\theta C(Y) \Rightarrow e^*\text{-}cl_\theta(f[U]) = f[U] \subseteq f[cl(U)].$$

Since f is bijective, then by Theorem 2.8(e) f is weakly e^*-θ-open. □

Theorem 2.22. *Let $f : X \to Y$ be a bijective function. If $f[cl_\theta(U)]$ is e^*-θ-closed in Y for every subset U of X, then f is weakly e^*-θ-open.*

Proof. Let $U \subseteq X$.

$$\left.\begin{array}{r} U \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[cl_\theta(U)] \in e^*\theta C(Y) \Rightarrow e^*\text{-}cl_\theta(f[U]) \subseteq e^*\text{-}cl_\theta(f[cl_\theta(U)]) = f[cl_\theta(U)].$$

Since f is bijective, then by Theorem 2.8(i) f is weakly e^*-θ-open. □

Definition 2.23. A function $f : X \to Y$ is called complementary weakly e^*-θ-open (briefly $c.we^*\theta\text{-}o.$) if for each open set U of X, $f[Fr(U)]$ is e^*-θ-closed in Y, where $Fr(U)$ denotes the frontier of U.

Remark 2.24. The following examples show that the notions complementary weakly e^*-θ-openness and weakly e^*-θ-openness are independent.

Example 2.25. Let $X = \{a,b,c,d\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\}$. Define the function $f : (X,\tau) \to (X,\tau)$ by $f = \{(a,d),(b,b),(c,d),(d,d)\}$. Then, f is complementary weakly e^*-θ-open which is not weakly e^*-θ-open.

Example 2.26. Let $X = \{a,b,c\}$, $\tau = \{\varnothing, X, \{a\}, \{a,c\}\}$ and $Y = \{1,2,3,4\}$, $\tau^* = \{\varnothing, Y, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$. Define the function $f : (X,\tau) \to (Y,\tau^*)$ by $f = \{(a,1),(b,3),(c,4)\}$. Then, f is weakly e^*-θ-open which is not complementary weakly e^*-θ-open.

Theorem 2.27. *If $f : X \to Y$ is bijective weakly e^*-θ-open and c.w.$e^*\theta$-o., then f is e^*-θ-open.*

Proof. Let $U \in O(X,x)$.

$$\left.\begin{array}{r} U \in O(X,x) \\ f \text{ is weakly } e^*\text{-}\theta\text{-open} \end{array}\right\} \overset{\text{Theorem 2.7(i)}}{\Rightarrow} (\exists V \in e^*\theta O(Y, f(x)))(V \subseteq f[cl(U)])(x \notin Fr(U) = cl(U) \setminus U)$$

$$\Rightarrow \left.\begin{array}{r} (y = f(x) \notin f[Fr(U)])(y \in V \setminus f[Fr(U)])(V_y := V \setminus f[Fr(U)]) \\ f \text{ is } c.w.e^*\theta\text{-}o. \end{array}\right\} \Rightarrow V_y \in e^*\theta O(Y,y)$$

$$\Rightarrow \left.\begin{array}{r} y \notin f[Fr(U)] = f[cl(U) \setminus U] \overset{f \text{ is bijective}}{=} f[cl(U)] \setminus f[U] \Rightarrow y \in f[U] \\ f[U] = \bigcup\{V_y | (V_y \in e^*\theta O(Y))(y \in f[U])\} \end{array}\right\} \Rightarrow f[U] \in e^*\theta O(Y).$$ □

Definition 2.28. A space X is said to be $e^*\theta$-connected [5] if X cannot be expressed as the disjoint union of two nonempty e^*-θ-open sets.

Theorem 2.29. *If $f : X \to Y$ is a bijective weakly e^*-θ-open of a space X onto an $e^*\theta$-connected space Y, then X is connected.*

Proof. Let f be a bijective weakly e^*-θ-open of a space X onto an $e^*\theta$-connected space Y and suppose that X is not connected.

$$\left.\begin{array}{r} X \text{ is not connected} \Rightarrow (\exists U_1, U_2 \in O(X) \setminus \{\varnothing\})(U_1 \cap U_2 = \varnothing)(U_1 \cup U_2 = X) \\ f \text{ is a weakly } e^*\text{-}\theta\text{-open bijection} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (f[U_i] \in O(Y) \setminus \{\varnothing\})(\bigcap_i f[U_i] = \varnothing)(\bigcup_i f[U_i] = Y)(f[U_i] \subseteq e^*\text{-}int_\theta(f[cl(U_i)]) = e^*\text{-}int_\theta(f[U_i]))\ (i = 1,2)$$

$$\Rightarrow (f[U_i] \in O(Y) \setminus \{\varnothing\})(\bigcap_i f[U_i] = \varnothing)(\bigcup_i f[U_i] = Y)(f[U_i] = e^*\text{-}int_\theta(f[U_i]))\ (i = 1,2)$$

$$\Rightarrow (f[U_i] \in e^*\theta O(Y) \setminus \{\varnothing\})(\bigcap_i f[U_i] = \varnothing)(\bigcup_i f[U_i] = Y)\ (i = 1,2).$$

Then, Y is not $e^*\theta$-connected which is a contradiction. □

Definition 2.30. A space X is said to be hyperconnected [18] if every nonempty open subset of X is dense in X.

Theorem 2.31. *If X is a hyperconnected space, then a function $f : X \to Y$ is weakly e^*-θ-open if and only if $f[X]$ is e^*-θ-open in Y.*

Proof. Necessity : Let $U \in O(X) \setminus \{\varnothing\}$.

$$\left.\begin{array}{r} \left.\begin{array}{r} U \in O(X) \setminus \{\varnothing\} \\ X \text{ is hyperconnected} \end{array}\right\} \Rightarrow cl(U) = X \Rightarrow e^*\text{-}int_\theta(f[cl(U)]) = e^*\text{-}int_\theta(f[X]) \\ f \text{ is weakly } e^*\text{-}\theta\text{-open} \end{array}\right\} \Rightarrow$$

$$\Rightarrow f[U] \subseteq f[X] = e^*\text{-}int_\theta(f[X]) = e^*\text{-}int_\theta(f[cl(U)]).$$

Sufficiency : Obvious. □

Definition 2.32. Two nonempty subsets A and B in X are strongly separated [24], if there exist open sets U and V in X with $A \subseteq U$ and $B \subseteq V$ and $cl(U) \cap cl(V) = \varnothing$.

Definition 2.33. A topological space X is said to be $e^*\theta$-T_2 [4] if for every pair of distinct points x and y, there exist two $e^*\theta$-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Theorem 2.34. *If $f : X \to Y$ is weakly e^*-θ-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is $e^*\theta$-T_2.*

Proof. Let $y, z \in Y$ and $y \neq z$.

$$\left.\begin{array}{r}(y, z \in Y)(y \neq z)\\ f^{-1}(y) \text{ and } f^{-1}(z) \text{ are strongly separated}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in O(X, f^{-1}(y)))(\exists V \in O(Y, f^{-1}(z)))(cl(U) \cap cl(V) = \emptyset)\\ f \text{ is weakly } e^*\text{-}\theta\text{-closed}\end{array}\right\} \overset{\text{Theorem 2.13(e)}}{\Rightarrow}$$

$$\left.\begin{array}{r}\Rightarrow (\exists A \in e^*\theta O(Y, y))(\exists B \in e^*\theta O(Y, z))(f^{-1}[A] \cap f^{-1}[B] \subseteq cl(U) \cap cl(V) = \emptyset)\\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists A \in e^*\theta O(Y, y))(\exists B \in e^*\theta O(Y, z))(A \cap B = \emptyset).$$ □

Corollary 2.35. *If $f : X \to Y$ is weakly e^*-θ-closed surjection with all fibers closed and X is normal, then Y is $e^*\theta$-T_2.*

Proof. It is obvious since disjoint closed sets in a normal space are strongly separated. □

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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