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PAPERS

Observability, controllability and stability of a nonlinear RLC circuit in form of a Duffing oscillator by means of theoretical mechanical approach

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In this research article, observability, controllability and stability of a nonlinear RLC circuit with a nonlinear capacitor is investigated as a Duffing oscillator beginning with the dissipative equations of generalized motion using Lagrange-dissipative model ($\{L, D\}$-model briefly). The force related to the potential energy, equilibria, and their well known stability properties are given using state space approach. Prerequisite that the condition for a Legendre transform is fulfilled, for the same system, also Hamiltonian of the system is found. Using Hamiltonian and dissipation function, dissipative canonical equations are obtained. These equations are written in state space form. Then the equality to the same results obtained using the dissipative equations of generalized motion related equilibria and their stability was shown. Thus a Lyapunov function as residual energy function (REF) is justified in terms of stability of the overall system. As last step, different electrical and mechanical (physical) realization possibilities are discussed.

K e y w o r d s: observability, controllability, Duffing oscillator/equation, equations of generalized motion, dissipative canonical equations, Lyapunov stability function

1 Introduction

In order to examine an oscillating mechanical system, Georg Duffing introduced a nonlinear second order differential equation in 1918, [1]. The concomitant equation is named as the Duffing equation. The involved oscillation is called the Duffing oscillation.

As outlined in the literature, dynamic systems which are described by Duffing equation occurs virtually in every field of science. Duffing oscillations, naturally, exist not only in mechanical but also in engineering, chaotical, biological, *etc* systems.

There are a lot of books on nonlinear dynamical systems and chaos covering Duffing equation/oscillator like [2–5]. A special source as a book on the Duffing equation is [6]. Different solution and analysis and synthesis approachs are found for example in [7–10]. Mathematical and physical foundations of Lagrangians and Hamiltonians are covered in [11–13] where [13] is one of the very rare books including conditions for a Legendre transform. On the other hand, a physical or an engineering system can be modeled by means of a Lagrangian L, a generalized velocity proportional Rayleigh dissipation function D and a Hamiltonian H depending on tensorial variables in covariant and contravariant forms. This was shown in [14] that covers also the dissipative canonical equations. The extended Hamiltonians in different tensorial forms to obtain equations of generalized motion in case of dissipative systems directly are given in [15] which includes also higher order $\{L, D\}$-models. Accordingly, higher order Lagrangians, dissipation functions and nonconservative Hamiltonians are presented in this study. Practical realization of a nonlinear capacitor in a basic form is given for example in [16]. In the reference [17], $\{L, D\}$-model of the Duffing equation in form of a RLC circuit can be found, where the circuit contains a nonlinear capacitor, voltage-charge characteristics of which is an odd function of order three as given here. Another source on nonlinear capacitor with a voltage-charge characteristics of order five is [18]. The reference [19] is about obtaining Lyapunov functions as residual energy functions in a systematic and different way than the other approaches. [20] includes control of chaotic Duffing equation with uncertainty in all parameters. Observability and controllability of nonlinear systems in general are given in [21]. Observability using Lie derivatives and controllability using Lie brackets of nonlinear systems are explained mathematically in detail in the reference [22], where the approach converts to the usual linear case of controllability and observability matrices, when the system is a linear one. The sources [23] and [24] are referred in order to show the practical meaning and importance of nonlinear physical systems.

But observability and controllability analysis of Duffing equation/oscillator in general lacks in the literature. And this analyze is performed first time using theoretical mechanical approach including dissipation with state space method as far as we know. Additionally, Lyapunovs direct (or second) method is applied as residual energy function and this is not considered for Duffing equation/oscillator in any other literature before. This pro-

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vides more precise results from the point of physics. Other stability result given in some internet sources are not correct which can be proven easily.

In this article a nonlinear RLC circuit as in Fig. 1 with a nonlinear term, namely capacitor C is investigated as a Duffing oscillator within the framework of Lagrange-dissipative model. Assuming that Legendre transform is provided the Hamiltonian of the nonlinear RLC circuit is obtained. Dissipative canonical equations are derived out of Hamiltonian and dissipative function. As far as we know, for the first time system under consideration is analysed by means of observability, controllability and also stability using Lyapunov function which is valid for linear and nonlinear systems. On the other hand, Lyapunov function as a residual energy function achieved as sum of Hamiltonian and negative form of dissipative function has been used for the analysis of the stability. Finally, the conclusions that are established has been stated.

2 The equations of dissipative generalized motion

The RLC circuit with a linear resistance R, linear inductor L^*, nonlinear capacitor C and the driving electromotive force $E = E_0\cos(\omega t)$ is given below.

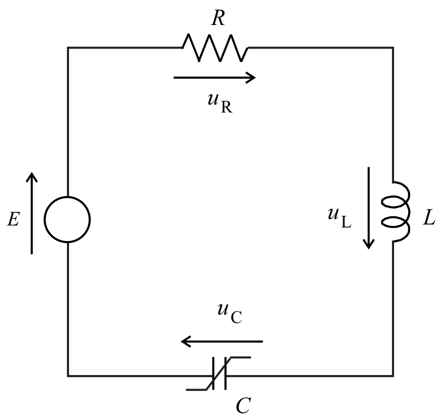

Fig. 1. RLC circuit with a nonlinear capacitor

A nonlinear capacitor can be realized by filling the space between the two conductor plates with a nonlinear ferroelectric material such as barium titanate. Then the voltage-charge characteristics will not be a straight line anymore. Assuming that the voltage-charge characteristics of this nonlinear capacitor C is approximated through an odd function of order three as $u_C = c_1q+c_2q^3$, where q is the charge, c_1 and c_2 are numeric constants. Applying Kirchoffs voltage law, one obtains

$$L^*\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + u_C = E \Rightarrow \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{R}{L^*}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{c_1}{L^*}q + \frac{c_2}{L^*}q^3 = \frac{E_0}{L^*}\,, \tag{1}$$

On the other hand, the most general form of Duffing oscillator has the form below

$$\ddot{x} + r\dot{x} + \omega_0^2x + \beta x^3 = \gamma\cos(\omega t)\,. \tag{2}$$

Althoug different definitions of Duffing equation (2) already exist (for example using $\omega_0^2 < 0$ in some sources on nonlinear differential equations), we prefer here the physical one with $\omega_0^2 > 0$ which converts to a harmonic oscillator with damping (without damping when $r = 0$) when $\beta = 0$.

Comparing (1) and (2), one can determine the generalized elements and the normalized generalized external force as follows

$$\begin{gathered}\ddot{x} + r\dot{x} + \omega_0^2x + \beta x^3 = \gamma\cos(\omega t)\,,\\ \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{L^*}(c_1q + c_2q^3) = \frac{E_0\cos(\omega t)}{L^*}\,,\\ r = \frac{R}{L^*},\ \omega_0^2 = \frac{c_1}{L^*},\ \beta = \frac{c_2}{L^*},\ \gamma = \frac{E_0}{L^*}\,,\end{gathered} \tag{3}$$

where, $L^*, R, c_1 \in \mathbb{R}^+$, $\omega t \neq (2n+1)\frac{\pi}{2}$, $n \in Z$. As can be seen again, when $c_2 = 0$, Duffing oscillator converts to a linear damped oscillator.

The generalized velocity proportional Rayleigh dissipation function for the case is

$$\frac{\partial D}{\partial \dot{q}} = \frac{R}{L^*}\dot{q} \Rightarrow D(\dot{q}) = \frac{R}{2L^*}\dot{q}^2. \tag{4}$$

Accordingly, the related $\{L, D\}$-model with an autonomous Lagrangian L, the related momentum and its first time derivative have the form given by

$$\left.\begin{aligned} L &= \underbrace{\tfrac{L^*}{2}\dot{q}^2}_{T(\dot{q})} - \underbrace{\left(\tfrac{c_1}{2L^*}q^2 + \tfrac{c_2}{4L^*}q^4\right)}_{U(q)}\\ D(\dot{q}) &= \tfrac{R}{2L^*}\dot{q}^2\end{aligned}\right. \Rightarrow \begin{aligned} p &= L^*\dot{q}\\ \dot{p} &= -\left(\tfrac{c_1}{L^*}q + \tfrac{c_2}{L^*}q^3\right)\end{aligned}, \tag{5}$$

where the external generalized force $E_0\cos(\omega t)$ may be included in the generalized velocity proportional (Rayleigh) dissipation function $D(\dot{q})i$ in form $\dot{q}\gamma\cos(\omega t)a$ as negative loss or in the Lagrangian as negative potential in form of $-q\gamma\cos(\omega t)$ to obtain the equations of generalized motion using extended Euler-Lagrange differential equation: $T(\dot{q})i$ is the kinetic and $U(q)$ is the potential energy parts of the Lagrangian.

Using the $\{L, D\}$-model of the system, the differential equation (1) can be obtained through the extended Euler-Lagrange differential equation which is given below

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = F_k\,,\ k = 1, 2, \ldots, f\,, \tag{6}$$

where k is the degree of freedom.

The force related potential energy of the Duffing oscillator via its $\{L, D\}$-model is obtained as follows. The force on nonlinear capacitive element related with the potential energy given above is

$$F_q = -\nabla U(q) \equiv -\operatorname{grad} U(q) = -\left(\frac{c_1}{L^*}q + \frac{c_2}{L^*}q^3\right)\,, \tag{7}$$

which is equal to the first time derivative of the generalized momentum as seen.

3 State space equations of the Duffing oscillator

Renaming the generalized coordinates and velocities as follows

$$q(t) = q_1(t)\,,\ \dot q_1(t) = q_2(t)\,. \tag{8}$$

The differential equation of dissipative generalized motion in (1) for the case takes the following form and can be rewritten as

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\dot q} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c_1+c_2q_1^2}{L^*} & -\frac{R}{L^*} \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{q} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L^*} \end{bmatrix} E_0\cos(\omega t)}_{F_e}\,,\ \underbrace{y = q_1}_{h(q)}\,, \tag{9a}$$

$$= \underbrace{\begin{bmatrix} q_2 \\ -\frac{c_1q_1+q_1^3+Rq_2}{L^*} \end{bmatrix}}_{f(q)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L^*} \end{bmatrix} E_0\cos(\omega t)}_{g(q)u}\,. \tag{9b}$$

3.1 Observability and controllability of the oscillator

To prove the nonlinear observability and controllability of the Duffing oscillator, the following general system in the control affine form will be used

$$\dot q = f(q) + \sum_{i=1}^{m} g_i(q)u_i\,,\quad y_i = h_i(q)\,, \tag{10}$$

where $\boldsymbol{q} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^m$ is the input vector and $\boldsymbol{y} \in \mathbb{R}^p$, $p < n$ is the output vector. Further, $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ are to be smooth vector fields.

A nonlinear system is then locally observable in P_0, when the rank of the observability matrix $\boldsymbol{O}_s$ equals n, where the observation space O_s is defined as the gradient of the vector of all Lie derivatives $L_f^n h$. And the Lie derivatives of $\boldsymbol{h}$ with respect to $\boldsymbol{f}$ are

$$L_f^0\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{q})\,,\ L_f^1\boldsymbol{h} = \nabla\boldsymbol{h}(\boldsymbol{q})\cdot\boldsymbol{f}(\boldsymbol{q})\,,\ldots,\ L_f^n\boldsymbol{h} = \nabla[L_f^{n-1}\boldsymbol{h}]\cdot\boldsymbol{f}(\boldsymbol{q})\,. \tag{11}$$

Accordingly, this leads here

$$\nabla\boldsymbol{h}(\boldsymbol{q}) = [1\ \ 0] \Rightarrow \begin{cases} \boldsymbol{h}(\boldsymbol{q}) = q_1 \\ L_f^1\boldsymbol{h} = [1\ \ 0]\begin{bmatrix} q_2 \\ -\frac{c_1q_1+c_2q_1^3+Rq_2}{L^*} \end{bmatrix} = q_2\,. \end{cases} \tag{12}$$

The vector of all Lie derivatives and its multiplication with ∇ are

$$\mathbf{G} = [L_f^0\boldsymbol{h}\ \ L_f^1\boldsymbol{h}] \Rightarrow \nabla\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

And the determinant of which has rank two, *ie* the system is (locally) observable.

Now, let us examine the controllability of the Duffing oscillator

The system in (8) above is locally accessible about a point $P_0 \in \mathbb{R}^n$ if the accesibility distribution Q spans $\mathbb{R}^n$ space when n equals the rank of $\boldsymbol{q}$ and the accessibility distribution Q is defined

$$\boldsymbol{Q}\big[g_1, g_2, \ldots, g_m, [\boldsymbol{ad}_f^1, \boldsymbol{g}_1], \ldots, [\boldsymbol{ad}_f^1, \boldsymbol{g}_m], \ldots, [\boldsymbol{ad}_f^{m-1}, \boldsymbol{g}_m]\big]\,, \tag{14}$$

where $[\boldsymbol{ad}_{g_i}^k, \boldsymbol{g}_j]; k \in \mathbb{N}_0$ is higher order Lie bracket. Using the Lie bracket, one achieves

$$[\boldsymbol{f}, \boldsymbol{g}] = -\underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c_1+3c_2q_1^2}{L^*} & -\frac{R}{L^*} \end{bmatrix}}_{[J]} \begin{bmatrix} 0 \\ \frac{1}{L^*} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L^*} \\ \frac{R}{(L^*)^2} \end{bmatrix}\,, \tag{15}$$

where $[J]$ is the Jacobian of the matrix $[\boldsymbol{f}(q)]$. Accessibility distribution $\boldsymbol{Q}$ is

$$\boldsymbol{Q} = \big[\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]\big] = \begin{bmatrix} 0 & -\frac{1}{L^*} \\ \frac{1}{L^*} & \frac{R}{(L^*)^2} \end{bmatrix}\,, \tag{16}$$

and it has rank two everywhere, therefore the system is accessible and controllable.

4 Stability of the equilibrium points of the oscillator

Although stability investigation for Duffing oscillator using chacteristic equations are well known for a relatively long time, nevertheless they are given here in order to completeness.

This system can be linearized around equilibria. The equilibrium points in phase plane are achieved when $(q; \dot q) = (0; 0)$ and the external force $E_0\cos(\omega t)/L^* = 0$. Without the external force, the damped Duffing oscillator will end up at one of its stable equilibrium points and they are given in what follows

$$\underbrace{\frac{q}{L^*}(c_1 + c_2q^2)}_{\nabla U(q)} = 0\,. \tag{17}$$

The first root $q_{11} = 0$ is always an equilibrium point. The other equilibrium points can be found as in what follows

a) $c_1 > 0, c_2 > 0$ (hardening case)
   In this case the equilibrium points are conjugated complex in form of $q_{12,13} = \pm j\sqrt{c_1/c_2}$.

b) $c_1 > 0, c_2 < 0$ (softening case)
   The equilibria of this case are in the form of $q_{12,13} = \pm\sqrt{c_1/|c_2|}$.

Using Jacobian matrix $[J]$ of the system and its eigenvalues, stability of these equilibria may be better understandable. With the following characteristic equations, one can obtain the results:

$$|\lambda[I]-[J]| = \begin{vmatrix} \lambda & -1 \\ \frac{c_1+3c_2q_1^2}{L^*} & \lambda+\frac{R}{L^*} \end{vmatrix} = \lambda^2+\lambda\frac{R}{L^*}+\frac{c_1+3c_2q_1^2}{L^*}=0\,. \quad (18)$$

a) For the equilibrium $q_{11}=0$, the characteristic equation of the Jacobian matrix $J[A]$ related to the system is

$$\lambda^2+\frac{R}{L^*}\lambda+\frac{c_1}{L^*}=0\,. \quad (19)$$

If $c_1\in\mathbb{R}^+$, which is the case for us, then the eigenvalues are both negative:

$$\left.\begin{aligned}\lambda_1+\lambda_2&=-\frac{R}{L^*}\\ \lambda_1\lambda_2&=\frac{c_1}{L^*}\end{aligned}\right\}\Rightarrow \lambda_1<0\,,\ \lambda_2<0\,, \quad (20)$$

ie the equilibrium point $q_{11}=0$ is stable.

b) For the other equilibrium points in hardening case, $q_{12,13}=\pm j\sqrt{c_1/c_2}$ which are conjugated complex, the characteristic equation of the Jacobian matrix $[J]$ related to the system is

$$\lambda^2+\frac{R}{L^*}-\frac{2c_1}{L^*}=0\,. \quad (21)$$

Herewith the sign of the roots are as follows

$$\left.\begin{aligned}\lambda_1+\lambda_2&=-\frac{R}{L^*}\\ \lambda_1\lambda_2&=-\frac{2c_1}{L^*}\end{aligned}\right\}\Rightarrow \lambda_1>0\,,\ \lambda_2<0\,. \quad (22)$$

Therefore, the equilibrium points $q_{12,13}=\pm\sqrt{c_1/c_2}$ are unstable, phase portraits of which are each in the form of a saddle point, where the trajectories are hyperbola like curves having the vertical and horizontal axes as asymptotes.

c) In softening case, the equilibria are real in form of $q_{12,13}=\pm\sqrt{c_1/|c_2|}$ and the case is just like in the hardening case and it will not be given again. Relating the equilibria, and their stability, see Fig. 2.

5 The Legendre transform and the Hamiltonian of the system

Prerequisite to that the condition for a Legendre transform given below

$$\Big|\frac{\partial^2 L}{\partial\dot q_j\partial\dot q_k}\Big|=\Big|\frac{\partial p_j}{\partial\dot q_k}\Big|=L^*\neq 0\,,\ j,k=1,2,\dots,f\,, \quad (23)$$

that is already fulfilled here, then the Hamiltonian $\mathcal{H}$ and through this the extended Hamiltonian $\boldsymbol{H}$ can be determined as follows

$$\boldsymbol{H}=\underbrace{\frac{p^2}{2L^*}+\Big(\frac{c_1}{2L^*}q^2+\frac{c_2}{4L^*}q^4\Big)}_{H(p,q)}+\frac{R}{L^*}q\dot q\,, \quad (24)$$

where the extended Hamiltonian is obtained for a system with f degree of freedom using

$$\boldsymbol{H}=\sum_{k=1}^{f}\frac{\partial L}{\partial\dot q_k}-L+\sum_{k=1}^{f}\frac{\partial D}{\partial\dot q_k}q_k\,. \quad (25)$$

6 Dissipative canonical equations and equilibria of the system

The dissipative canonical equation of this lossy system is given by

$$\dot q=\frac{\partial\boldsymbol{H}}{\partial p}=\frac{p}{L^*}\,, \quad (26a)$$

$$\dot p=-\frac{\partial\boldsymbol{H}}{\partial q}=-\frac{c_1}{L^*}q-\frac{c_2}{L^*}q^3-\frac{R}{L^*}\dot q\,. \quad (26b)$$

Equilibria condition for such (dissipative) Hamiltonian systems in general is

$$(\dot q_k\,;\,\dot p_k)=\Big(\frac{\partial\boldsymbol{H}}{\partial p_k}\,;\,\frac{\partial\boldsymbol{H}}{\partial q_k}\Big)=(0\,;\,0)\,. \quad (27)$$

For our case, this is as follows

$$\frac{\partial\boldsymbol{H}}{\partial p}=0\Rightarrow p=0\,,\ \frac{\partial\boldsymbol{H}}{\partial q}=0\Rightarrow c_1q+c_2q^3=-R\dot q\,. \quad (28)$$

The right part of the equation above is a nonlinear homogenious differential equation of order one and as can be seen, normally dissipative part of the equations cannot be ignored when finding equilibrium points for dissipative Hamiltonian systems. But here one finds out that among generalized momentum and generalized velocity, a simple relation in the following form already exists

$$p=L^*\dot q=0 \text{ where } \dot q=0\ \forall L^*\in\mathbb{R}^+\,, \quad (29)$$

which leads to the same equilibria and roots as in the case $\nabla U(q)=0$.

Moreover, (26) can be rewritten in form of a matrix equation including normalized external generalized force(s) as follows

$$\frac{\partial}{\partial t}\begin{bmatrix}q\\p\end{bmatrix}=\underbrace{\frac{1}{L^*}\begin{bmatrix}0&1\\-c_1-c_2q^2&-R\end{bmatrix}}_{\boldsymbol{A}}\begin{bmatrix}q\\p\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{L^*}\end{bmatrix}E_0\cos(\omega t)\,, \quad (30)$$

where $\boldsymbol{A}$ is the system matrix as before. Similarities among state space form (9a) obtained through equations of dissipative generalized motion and state space form (30) obtained through dissipative canonical equations are obvious. Eigenvalues for stability have also the same properties and will not be presented again.

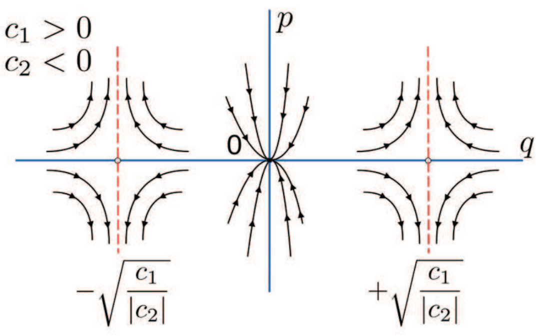

Fig. 2. The equilibria and their stability

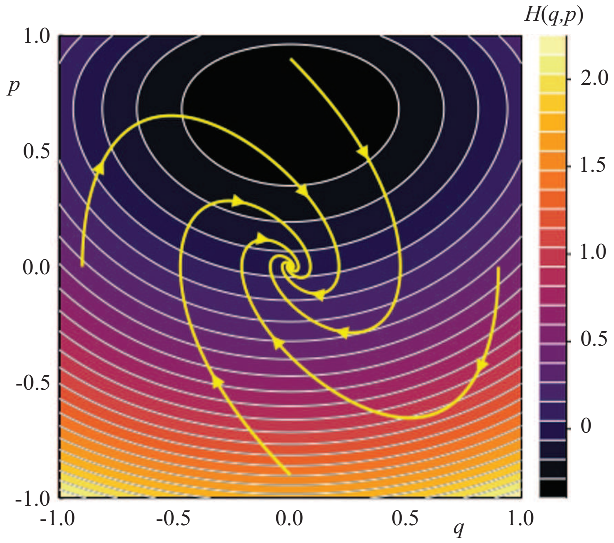

Fig. 3. The trajectories in phase space for marginal and asymptotic stability

6 Lyapunov function as residual energy function and stability of the system

The residual energy function for this case is

$$\boldsymbol{H} = \frac{p^2}{2L^*} + \left(\frac{c_1}{2L^*}q^2 + \frac{c_2}{4L^*}q^4\right) - \int \left[\frac{R}{2(L^*)^2}p^2\right] \mathrm{d}t \,. \quad (31)$$

Since the REF is defined as below

$$\boldsymbol{H} = H^+ - \int \Big[\sum_{k=1}^{f} \frac{\partial D(\dot q_1, \dot q_2, \dots, \dot q_f)}{\partial \dot q_k} \dot q_k\Big] \mathrm{d}t \,,$$

$$0 < H < \infty \,, \ \forall t \in \mathbb{R}_0^+ \,, \quad (32)$$

and the first time derivation of this REF, which is total power, is

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = -\sum_{k=1}^{f} \frac{\partial D(\dot q_1, \dot q_2, \dots, \dot q_f)}{\partial \dot q_k} \dot q_k \,. \quad (33)$$

The REF of this dissipative system is

$$H = \Big[\underbrace{\frac{p^2}{2L^*}}_{T(p)} + \underbrace{\left(\frac{c_1}{2L^*}q^2 + \frac{c_2}{4L^*}q^4\right)}_{U(q)}\Big] - \frac{R}{L^*}\int \dot q \,\mathrm{d}t \,, \quad (34)$$

with the first time derivative

$$\dot H(p_k, q^k) = -\frac{R}{L^*}\dot q \,. \quad (35)$$

This shows that REF fulfills all the properties of a Lyapunov function, For stability of the system, the necessary condition is given in what follows

$$-\frac{R}{L^*} \le 0 \,. \quad (36)$$

That is, it must be positive semidefinite. And as such the origin point $(q;p) = (0;0)$ in phase space is marginal or asymptotically stable and the trajectories in phase space are either in form of ellipses around origin or spirals toward the origin, as shown in the Fig. 3.

Besides as in the example (charge formulation in which the generalized coordinate q is the charge), such a physical system can be realized in different ways. Three of them are given below:

a) Using flux formulation (Ψ -is flux): a paralel nonlinear RLC circuit in which the current through the nonlinear inductive element L is i_L and the generalized external force is the current source $I = I_L$.
b) Using translational mechanical (or displacement/position) formulation (x is translational displacement/position): a dissipative harmonic oscillator consisting of a mass m, a damper D and a spring k in which the force on the nonlinear spring k is $F_k = c_k x + d_k x^3$ and the generalized external force is the force $F = F_x \cos(\omega t)$.
c) Using rotational mechanical (or angular) formulation (θ is angular displacement): a dissipative rotational harmonic oscillator consisting of a moment of inertia I, a rotational damper D_r and a torsional spring k_r in which the torque on the nonlinear torsional spring k_r in which the torque on the nonlinear torsional spring k_r is $M_r = c_r\theta + d_r\theta^3$ and the generalized external force is the torque $M = M_\theta \cos(\omega t)$.

7 Conclusions

In conclusion in a systematic manner it was demonstrated that for Duffing oscillator/equation as a nonlinear system in form of a nonlinear RLC circuit containing a nonlinear capacitor, when {L,D}-model and thus Hamiltonian are known, then the system is analysed by means of observability, controllability and stability in this fromount which was performed first time for Duffing oscillator/equation in general here. Moreover, stability analysis can be performed using Lyapunov function as residual energy function. As seen, this kind of Lyapunov function can be constructed using Hamiltonian and dissipative function together for linear and nonlinear systems as well.

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