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On characterizations of weakly e^*-continuity

Sobre caracterizaciones de la e^*-continuidad débil

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Abstract

The aim of this paper is to investigate some of the fundamental properties of weakly e^*-continuous functions introduced by Ayhan and Özkoç in [3]. Moreover, several characterizations and some properties concerning weakly e^*-continuous functions are obtained. Then, we investigate relationships between weak e^*-continuity and some other types of continuity. Also, we investigate the relationships between weakly e^*-continuous functions and connectedness and graph of functions.

Key words and phrases: e^*-open, weakly e^*-continuity, almost e^*-continuity, faintly e^*-continuity.

Resumen

El objetivo de este trabajo es investigar algunas de las propiedades fundamentales de las funciones e^*-continuas débilmente introducidas por Ayhan y Özkoç en [3]. Además, se obtienen varias caracterizaciones y algunas propiedades relativas a funciones e^*-continuas débilmente. Luego, investigamos las relaciones entre la dbil e^*-continuas débil y algunos otros tipos de continuidad. Además, investigamos las relaciones entre las funciones e^*-continuas débilmente y la conectividad y gráfica de funciones.

Palabras y frases clave: e^*-abierto, e^*-continuidad débil, casi e^*-continuidad, e^*-continuidad ligera.

1 Introduction

One of the most important subjects in mathematics is the notion of continuity. Recently, several studies have been carried out on continuous functions which are indispensable objects of topology. In these studies, the concepts which are both stronger and weaker than the concept of continuity

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have been introduced and some fundamental results have been obtained. For instance in 1961, the concept of weak continuity is introduced by Levine in [8]. In the following years various weak forms of weak continuity were defined and studied by many mathematicians. The concept of weak e^*-continuity which is defined by Ayhan and Özkoç in [3], weaker than weak e-continuity introduced by Özkoç and Aslım in [12], weak β-continuity introduced by Popa and Noiri in [14], weak b-continuity introduced by Şengül in [15], almost e^*-continuity and weak a-continuity introduced by Ayhan and Özkoç in [3], but stronger than faint e^*-continuity introduced by Jafari and Rajesh in [7]. In this paper, we study weak e^*-continuity via e^*-open sets defined by Ekici in [5].

2 Preliminaries

Throughout this present paper, (X,τ) and (Y,σ) (or simply X and Y) represent topological spaces on which no seperation axioms are assumed. For a subset A of a space X, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. The family of all open (resp. closed, clopen) sets of a topological space X will be denoted by $O(X)$ (resp. $C(X)$, $CO(X)$). A subset A of a topological space X is said to be regular open (regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$) (cf. [16]). The family of all regular open (regular closed) of a topological space X is denoted by RO(X) (RC(X)). A point x of X is said to be δ-cluster point of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x (cf. [17]). The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $cl_\delta(A)$ (cf. [17]). If $A = cl_\delta(A)$, then A is called δ-closed, and the complement of a δ-closed set is called δ-open (cf. [17]). The set $\{x : \exists\ U \in RO(X) \text{ with } x \in U \subseteq A\}$ (equally $\{x : \exists\ U \in \tau \text{ with } x \in U \text{ and } int(cl(U)) \subseteq A\}$) is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called a-open (resp. semiopen, preopen, b-open, β-open, e-open, e^*-open) if $A \subseteq int(cl(int_\delta(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int(cl(A)))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subseteq cl(int(cl_\delta(A)))$) (cf. [4, 9, 10, 2, 1, 6, 5] respectively). The complement of an a-open (resp. semiopen, preopen, b-open, β-open, e-open, e^*-open) set is called a-closed (resp. semiclosed, preclosed, b-closed, β-closed, e-closed, e^*-closed), see [4, 9, 10, 2, 1, 6, 5] respectively. The intersection of all e^*-closed sets of X containing A is called the e^*-closure of A and is denoted by e^*-$cl(A)$ (cf. [5]). The union of all e^*-open sets of X contained in A is called the e^*-interior of A and is denoted by e^*-$int(A)$ (cf. [5]).

A point x of X is called a θ-cluster point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x (cf. [17]). The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $cl_\theta(A)$ (cf. [17]). Equivalently $cl_\theta(A) = \bigcap\{F : A \subseteq int(F) \text{ and } F \in C(X)\}$. A subset A is said to be θ-closed if $A = cl_\theta(A)$ (cf. [17]). The complement of a θ-closed set is called a θ-open set (cf. [17]). A point x of X said to be a θ-interior point of a subset A, denoted by $int_\theta(A)$, if there exists an open set U of X containing x such that $cl(U) \subseteq A$ (cf. [17]). Equivalently $int_\theta(A) = \bigcup\{U : cl(U) \subseteq A \text{ and } U \in O(X)\}$.

The family of all open (resp. closed, e-open, e-closed, e^*-open, e^*-closed, β-open, β-closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, a-open, a-closed) subsets of X is denoted by $O(X)$ (resp. $C(X)$, $eO(X)$, $eC(X)$, $e^*O(X)$, $e^*C(X)$, $\beta O(X)$, $\beta C(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $PC(X)$, $aO(X)$, $aC(X)$). The family of all open (resp. closed, e^*-open, e^*-closed, β-open, β-closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, a-open, a-closed) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $eO(X,x)$, $eC(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $\beta O(X,x)$,

$\beta C(X,x)$, $\delta O(X,x)$, $\delta C(X,x)$, $\theta O(X,x)$, $\theta C(X,x)$, $SO(X,x)$, $SC(X,x)$, $PO(X,x)$, $PC(X,x)$, $aO(X,x)$, $aC(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

The following basic properties of e^*-closure and e^*-interior are useful in the sequel:

Lemma 2.1 (cf. [5]). *Let A be a subset of a space X, then the followings hold:*

1. $e^*\text{-}cl(X \setminus A) = X \setminus e^*\text{-}int(A)$.
2. $x \in e^*\text{-}cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in e^*O(X,x)$.
3. A *is* $e^*C(X)$ *if and only if* $A = e^*\text{-}cl(A)$.
4. $e^*\text{-}cl(A) \in e^*C(X)$.
5. $e^*\text{-}int(A) = A \cap cl(int(cl_\delta(A)))$.

Lemma 2.2. *Let X be a topological space and $A \subseteq X$. Then the following properties hold:*

1. *If A is an open set in X, then* $cl(A) = cl_\delta(A) = cl_\theta(A)$.
2. $cl_\theta(A) \in C(X)$.

Proof.

1. Let $A \in O(X)$ and $x \in cl_\theta(A)$, then for all $U \in O(X,x)$ is obtained $cl(U) \cap A \neq \emptyset$. Therefore, for all $U \in O(X,x))$ we get that $cl(U \cap A) \supseteq cl(U) \cap A \neq \emptyset$, because $A \in O(X)$. So, for all $U \in O(X,x)$, $U \cap A \neq \emptyset$ and, therefore, $x \in cl(A)$. Then we have
$$cl_\theta(A) \subseteq cl(A) \tag{1}$$
On the other hand, we have always
$$cl(A) \subseteq cl_\delta(A) \subseteq cl_\theta(A) \tag{2}$$
By equations (1) and (2) we get that $cl(A) = cl_\delta(A) = cl_\theta(A)$.
2. It is obvious from the fact that $cl_\theta(A) = \bigcap\{F : A \subseteq int(F) \text{ and } F \in C(X)\}$.

□

Lemma 2.3 (cf. [3]). *Let X be a topological space and $A, B \subseteq X$. If A is an a-open set and B is an e^*-open set, then $A \cap B$ is an e^*-open set in X.*

Lemma 2.4 (cf. [13]). *Let X and Y two topological spaces and $A \subseteq X$ and $B \subseteq Y$. Then*
$$cl_\delta(A \times B) = cl_\delta(A) \times cl_\delta(B).$$

Definition 2.1. A function $f : X \to Y$ is said to be:

1. δ-continuous if $f^{-1}[V]$ is δ-open in X for each δ-open set V of Y (cf. [11]).
2. β-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta O(X,x)$ such that $f[U] \subseteq V$ (cf. [1]).

3. e^*-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq V$ (cf. [5]).

4. Weakly b-continuous (briefly w.b.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f[U] \subseteq cl(V)$ (cf. [15]).

5. Weakly β-continuous (briefly w.β.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta O(X, x)$ such that $f[U] \subseteq cl(V)$ (cf. [14]).

6. Weakly e-continuous (briefly w.e.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in eO(X, x)$ such that $f[U] \subseteq cl(V)$ (cf. [12]).

7. Weakly a-continuous (briefly w.a.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in aO(X, x)$ such that $f[U] \subseteq cl(V)$ (cf. [3]).

8. Almost e^*-continuous (briefly a.e^*.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq int(cl(V))$ (cf. [3]).

9. Faintly e^*-continuous (briefly f.e^*.c.) if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq V$ (cf. [3]).

Lemma 2.5 (cf. [11]). *Let $f : X \to Y$ be a function. Then the function is δ-continuous if and only if $f^{-1}[cl_\delta(A)] \subseteq cl_\delta(f^{-1}[A])$ for each subset A of X.*

3 Weakly e^*-continuous functions

Definition 3.1. Let X and Y be topological spaces. $f : X \to Y$ is a weakly e^*-continuous (briefly w.e^*.c.) at $x \in X$ if for each open set V containing $f(x)$, there exists an e^*-open set U in X containing x such that $f[U] \subseteq cl(V)$ (cf. [3]). The function f is w.e^*.c. if and only if f is w.e^*.c. for all $x \in X$.

Remark 3.1. From Definition 3.1 and Definition 2.1, we have the following diagram. The converses of these implications are not true in general, as shown in the following examples. Also, examples for the other implications are shown in the related papers.

$$\begin{array}{ccccc}
\text{weakly } b\text{-continuity} & \longrightarrow & \text{weakly } \beta\text{-continuity} & \longleftarrow & \beta\text{-continuity} \\
 & & \downarrow & & \downarrow \\
\text{weakly } e\text{-continuity} & \longrightarrow & \text{weakly } e^*\text{-continuity} & \longleftarrow & e^*\text{-continuity} \\
 & \nearrow & \downarrow & \nwarrow & \\
\text{almost } e^*\text{-continuity} & & \text{faintly } e^*\text{-continuity} & & \text{weakly } a\text{-continuity}
\end{array}$$

Examples 3.1.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. It is not difficult to see $e^*O(X) = 2^X \setminus \{\{d\}\}$. Define the function $f : X \to X$ by $f = \{(a, d), (b, b), (c, c), (d, a)\}$. Then f is weakly e^*-continuous but it is not e^*-continuous.

2. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. It is not difficult to see $eO(X) = 2^X \setminus \{\{d\}, \{a, d\}, \{c, d\}\}$ and $e^*O(X) = 2^X \setminus \{\{d\}\}$. Define the function $f : X \to X$ by $f = \{(a, b), (b, a), (c, c), (d, d)\}$. Then f is faintly e^*-continuous but it is not weakly e^*-continuous.

3. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. It is not difficult to see $eO(X) = 2^X \setminus \{\{d\}, \{a, d\}, \{c, d\}\}$ and $e^*O(X) = 2^X \setminus \{\{d\}\}$. Define the function $f : X \to X$ by $f = \{(a, d), (b, b), (c, c), (d, a)\}$. Then f is weakly e^*-continuous but it is not weakly e-continuous.

4. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\}$. It is not difficult to see $\beta O(X) = 2^X \setminus \{\{c\}, \{d\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $e^*O(X) = 2^X$. Define the function $f : X \to X$ by $f = \{(a, d), (b, b), (c, c), (d, a)\}$. Then f is weakly e^*-continuous but it is not weakly β-continuous.

Theorem 3.1. *The following properties are equivalent for a function $f : X \to Y$:*

1. *f is weakly e^*-continuous at $x \in X$.*
2. *For each neighbourhood V of $f(x)$, $x \in cl(int(cl_\delta(f^{-1}[cl(V)])))$.*
3. *For each neighbourhood V of $f(x)$ and each neighbourhood U of x, there exists a nonempty open set $G \subseteq U$ such that $G \subseteq cl_\delta(f^{-1}[cl(V)])$.*
4. *For each neighbourhood V of $f(x)$, there exists $U \in SO(X, x)$ such that $U \subseteq cl_\delta(f^{-1}[cl(V)])$.*

Proof.

$\boxed{1. \Rightarrow 2.}$ Let $V \in O(Y, f(x))$. By item *1.* we get that there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq cl(V)$. So, there exists $U \in e^*O(X, x)$ such that $U \subseteq f^{-1}[cl(V)]$. Then, there exists $U \in e^*O(X, x)$ such that $U \subseteq cl(int(cl_\delta(U))) \subseteq cl(int(cl_\delta(f^{-1}[cl(V)])))$. Therefore $x \in cl(int(cl_\delta(f^{-1}[cl(V)])))$.

$\boxed{2. \Rightarrow 3.}$ Let $V \in O(Y, f(x))$, then $x \in cl(int(cl_\delta(f^{-1}[cl(V)])))$, by item *2.*. Now, let $U \in O(X, x)$, then $x \in U \cap cl(int(cl_\delta(f^{-1}[cl(V)])))$. So, $U \cap cl(int(cl_\delta(f^{-1}[cl(V)]))) \neq \emptyset$ and, therefore, $cl\left(U \cap int(cl_\delta(f^{-1}[cl(V)]))\right) \neq \emptyset$. This implies that $U \cap int(cl_\delta(f^{-1}[cl(V)])) \neq \emptyset$. Define now $G := U \cap int(cl_\delta(f^{-1}[cl(V)]))$, then $G \in \tau \setminus \{\emptyset\}$, $G \subseteq U$ and

$$G \subseteq int(cl_\delta(f^{-1}[cl(V)])) \subseteq cl_\delta(f^{-1}[cl(V)]).$$

$\boxed{3. \Rightarrow 4.}$ Let $V \in O(Y, f(x))$ and $U \in O(X, x)$, then by item *3.* we get there exists $G_U \in \tau \setminus \{\emptyset\}$ such that $G_U \subseteq U$ and $G_U \subseteq cl_\delta(f^{-1}[cl(V)])$. Define now $G := \bigcup\{G_U | U \in O(X, x)\}$, then $G \in \tau$, $x \in cl(G)$ and $G \subseteq cl_\delta(f^{-1}[cl(V)])$. Also, defining $U_0 := G \cup \{x\}$, we get that $G \subseteq U_0 \subseteq cl(G)$ and $U_0 \subseteq cl_\delta(f^{-1}[cl(V)])$. Therefore $U_0 \in SO(X, x)$ and $U_0 \subseteq cl_\delta(f^{-1}[cl(V)])$.

$\boxed{4. \Rightarrow 1.}$ Let $V \in O(Y, f(x))$. By item *4.* we get that $x \in f^{-1}[V]$ and there exists $G \in SO(X, x)$ such that $G \subseteq cl_\delta(f^{-1}[cl(V)])$ and, therefore

$$x \in f^{-1}[V] \cap G \subseteq f^{-1}[cl(V)] \cap cl(int(G)) \subseteq f^{-1}[cl(V)] \cap cl(int(cl_\delta(f^{-1}[cl(V)])))$$

Now defining $U := e^*\text{-}int(f^{-1}[cl(V)]) = f^{-1}[cl(V)] \cap cl(int(cl_\delta(f^{-1}[cl(V)])))$, then $U \in e^*O(X, x)$ and $f[U] \subseteq cl(V)$. □

Theorem 3.2. *The following properties are equivalent for a function $f : X \to Y$:*

1. *f is weakly e^*-continuous.*
2. *$e^*\text{-}cl(f^{-1}[int(cl(B))]) \subseteq f^{-1}[cl(B)]$ for every subset B of Y.*

3. *e^*-$cl(f^{-1}[int(F)]) \subseteq f^{-1}[F]$ for every regular closed set F of Y.*
4. *$f^{-1}[B] \subseteq e^*$-$int(f^{-1}[cl(B)])$ for every regular open set B of Y.*
5. *e^*-$cl(f^{-1}[V]) \subseteq f^{-1}[cl(V)]$ for every open set V of Y.*
6. *$f^{-1}[int(F)] \subseteq e^*$-$int(f^{-1}[F])$ for every closed set F of Y.*
7. *$f^{-1}[V] \subseteq e^*$-$int(f^{-1}[cl(V)])$ for every open set V of Y.*
8. *e^*-$cl(f^{-1}[int(F)]) \subseteq f^{-1}[F]$ for every closed set F of Y.*
9. *$f^{-1}[V] \subseteq cl(int(cl_\delta(f^{-1}[cl(V)])))$ for every open set V of Y.*
10. *$int(cl(int_\delta(f^{-1}[int(F)]))) \subseteq f^{-1}[F]$ for every closed set F of Y.*

Proof.

$\boxed{1. \Rightarrow 2.}$ Let $B \subseteq Y$ and $x \notin f^{-1}[cl(B)]$, then $f(x) \notin cl(B)$ and there exists $V \in O(Y, f(x))$ such that $V \cap B = \emptyset$. So, there exists $V \in O(Y, f(x))$ such that $cl(V) \cap int(cl(B)) = \emptyset$. Also, by item *1.*, there exists $U \in e^*O(X, x)$ such that $f[U] \cap int(cl(B)) \subseteq cl(V) \cap int(cl(B))$. So, there exists $U \in e^*O(X, x)$ such that $U \cap f^{-1}[int(cl(B))] = \emptyset$ and, therefore $x \notin e^*$-$cl(f^{-1}[int(cl(B))]$.

$\boxed{2. \Rightarrow 3.}$ Let $F \in RC(Y)$, then $F \in C(Y)$ and therefore $F = cl(F)$. Now, by item *2.*, we get

$$e^*\text{-}cl(f^{-1}[int(F)]) = e^*\text{-}cl(f^{-1}[int(cl(F))]) \subseteq f^{-1}[cl(F)] = f^{-1}[F].$$

$\boxed{3. \Rightarrow 4.}$ Straightforward.

$\boxed{4. \Rightarrow 5.}$ Let $V \in O(Y)$, therefore $Y \setminus cl(V) \in RO(Y)$. So, by item *4.*, we get

$$f^{-1}[Y \setminus cl(V)] \subseteq e^*\text{-}int(f^{-1}[cl(Y \setminus cl(V))])$$

Then, $X \setminus f^{-1}[cl(V)] \subseteq X \setminus e^*$-$cl(f^{-1}[int(cl(V))])$ and, therefore

$$e^*\text{-}cl(f^{-1}[V]) \subseteq e^*\text{-}cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)].$$

$\boxed{5. \Rightarrow 6.}$ Straightforward.

$\boxed{6. \Rightarrow 7.}$ Let $V \in O(Y)$, then $cl(V) \in C(Y)$. So, by item *6.*, we get

$$f^{-1}[V] \subseteq f^{-1}[int(cl(V))] \subseteq e^*\text{-}int(f^{-1}[cl(V)]).$$

$\boxed{7. \Rightarrow 8.}$ Straightforward.

$\boxed{8. \Rightarrow 9.}$ Let $V \in O(Y)$, then $Y \setminus V \in C(Y)$. So, by item *8.*, we get that

$$e^*\text{-}cl(f^{-1}[int(Y \setminus V)]) \subseteq f^{-1}[Y \setminus V]$$

Then, $X \setminus e^*$-$int(f^{-1}[cl(V)]) \subseteq X \setminus f^{-1}[V]$ and

$$f^{-1}[V] \subseteq e^*\text{-}int(f^{-1}[cl(V)]) = f^{-1}[cl(V)] \cap cl(int(cl_\delta(f^{-1}[cl(V)])))$$

Therefore, $f^{-1}[V] \subseteq cl(int(cl_\delta(f^{-1}[cl(V)])))$.

$\boxed{9. \Rightarrow 10.}$ Straightforward.

$\boxed{10. \Rightarrow 1.}$ Let $x \in X$ and $V \in O(Y, f(x))$, then $Y \setminus V \in C(Y)$ and $x \in f^{-1}[V]$. Now, by item *10.*, we get that $int(cl(int_\delta(f^{-1}[int(Y \setminus V)]))) \subseteq f^{-1}[Y \setminus V]$. So, $f^{-1}[V] \subseteq cl(int(cl_\delta(f^{-1}[cl(V)])))$ and, therefore, $x \in cl(int(cl_\delta(f^{-1}[cl(V)])))$. □

Theorem 3.3. *The following properties are equivalent for a function $f : X \to Y$:*

1. *f is weakly e^*-continuous.*
2. *e^*-$cl(f^{-1}[V]) \subseteq f^{-1}[cl(V)]$ for each $V \in PO(Y)$.*
3. *$f^{-1}[int(F)] \subseteq e^*$-$int(f^{-1}[F])$ for each $F \in PC(Y)$.*
4. *$f^{-1}[V] \subseteq e^*$-$int(f^{-1}[cl(V)])$ for each $V \in PO(Y)$.*
5. *e^*-$cl(f^{-1}[int(F)]) \subseteq f^{-1}[F]$ for each $F \in PC(Y)$.*

Proof.

$\boxed{1. \Rightarrow 2.}$ Let $V \in PO(Y)$ and $x \notin f^{-1}[cl(V)]$, then $f(x) \notin cl(V)$ and there exists $W \in O(Y, f(x))$ such that $V \cap W = \emptyset$. So,

$$\begin{aligned} V \cap cl(W) &\subseteq int(cl(V)) \cap cl(W) \subseteq cl[int(cl(V)) \cap W] = cl[int(cl(V) \cap W)] \\ &\subseteq cl(int(cl(V \cap W))) \subseteq cl(V \cap W) = \emptyset \end{aligned}$$

By item *1.*, we get that there exists $U \in e^*O(X, x)$ such that $V \cap f[U] \subseteq V \cap cl(W)$. So, there exists $U \in e^*O(X, x)$ such that $f^{-1}[V] \cap U = \emptyset$ and, therefore, $x \notin e^*$-$cl(f^{-1}[V])$.

$\boxed{2. \Rightarrow 3.}$ Straightforward.

$\boxed{3. \Rightarrow 4.}$ Let $V \in PO(Y)$, then $cl(V) \in PC(Y)$ and $V \subseteq int(cl(V))$. Now, by item *3.*

$$f^{-1}[V] \subseteq f^{-1}[int(cl(V))] \subseteq e^*\text{-}int(f^{-1}[cl(V)]).$$

$\boxed{4. \Rightarrow 5.}$ Straightforward.

$\boxed{5. \Rightarrow 1.}$ This follows from item *6.* of Theorem 3.2, since every closed set is preclosed. □

Theorem 3.4. *The following properties are equivalent for a function $f : X \to Y$:*

1. *f is weakly e^*-continuous.*
2. *$f[e^*$-$cl(A)] \subseteq cl_\theta(f[A])$ for each subset A of X.*
3. *e^*-$cl(f^{-1}[B]) \subseteq f^{-1}[cl_\theta(B)]$ for each subset B of Y.*
4. *e^*-$cl(f^{-1}[int(cl_\theta(B))]) \subseteq f^{-1}[cl_\theta(B)]$ for each subset B of Y.*

Proof.

$\boxed{1. \Rightarrow 2.}$ Let $A \subseteq X$, $x \in e^*$-$cl(A)$ and $V \in O(Y, f(x))$. By item *1.*, we get that there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq cl(V)$. So, $U \cap A \neq \emptyset$ and $\emptyset \neq f[U] \cap f[A] \subseteq cl(V) \cap f[A]$. Therefore, $cl(V) \cap f[A] \neq \emptyset$. Then we have $f(x) \in cl_\theta(f[A])$.

$\boxed{2. \Rightarrow 3.}$ Let $B \subseteq Y$, so $f^{-1}[B] \subseteq X$. By item *2.*, we get that

$$f[e^*\text{-}cl(f^{-1}[B])] \subseteq cl_\theta(f[f^{-1}[B]]) \subseteq cl_\theta(B)$$

Then, $e^*\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[cl_\theta(B)]$.

$\boxed{3. \Rightarrow 4.}$ Let $B \subseteq Y$, then $int(cl_\theta(B)) \subseteq Y$. By item *3.* we get that

$$\begin{array}{rcl} e^*\text{-}cl(f^{-1}[int(cl_\theta(B))]) & \subseteq & f^{-1}[cl_\theta(int(cl_\theta(B)))] \\ & \overset{\text{Lemma 2.2(1)}}{=} & f^{-1}[cl(int(cl_\theta(B)))] \\ & \overset{\text{Lemma 2.2(2)}}{\subseteq} & f^{-1}[cl_\theta(B)]. \end{array}$$

$\boxed{4. \Rightarrow 1.}$ It is obvious from item *7.* from Theorem 3.2. □

Corollary 3.1. *Let $f : X \to Y$ be a function. If f is $w.e^*.c.$, then $f^{-1}[V]$ is e^*-closed in X for every θ-closed set V of Y.*

Proof. Let $V \in \theta C(Y)$, then $cl_\theta(V) = V$. Since f is $w.e^*.c.$ and, taking into consideration the item *3.* from Theorem 3.4, we get that $e^*\text{-}cl(f^{-1}[V]) \subseteq f^{-1}[cl_\theta(V)] = f^{-1}[V]$ and, therefore, $f^{-1}[V] \in e^*C(X)$. □

Corollary 3.2. *Let $f : X \to Y$ be a function. If $f^{-1}[cl_\theta(B)]$ is e^*-closed in X for every subset B of Y, then f is $w.e^*.c.$*

Proof. Let $B \subseteq Y$, by hypothesis we get that $f^{-1}[cl_\theta(B)] \in e^*C(X)$. So,

$$e^*\text{-}cl(f^{-1}[B]) \subseteq e^*\text{-}cl(f^{-1}[cl_\theta(B)]) = f^{-1}[cl_\theta(B)]$$

Then f is $w.e^*.c.$ by item *3.* from Theorem 3.4. □

Theorem 3.5. *Let X and Y be two topological spaces, and $f : X \to Y$ a function. If the graph function $g : X \to X \times Y$ of f, defined by $g(x) = (x, f(x))$ for each $x \in X$, is $w.e^*.c.$, then f is $w.e^*.c.$*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$, so $X \times V \in O(X \times Y, g(x))$. Since g is $w.e^*.c.$, then there exists $U \in e^*O(X, x)$ such that $g[U] \subseteq cl(X \times V) = X \times cl(V)$ and, therefore, there exists $U \in e^*O(X, x)$ such that $f[U] \subseteq cl(V)$. □

Corollary 3.3. *If in addition X is regular, then the converse of Theorem 3.5 is true.*

Proof. Let $x \in X$ and $W \in O(X \times Y, g(x))$, then there exists $U_1 \in O(X)$ and $V \in O(Y)$ such that $g(x) \in U_1 \times V \subseteq W$. Since f is $w.e^*.c.$, then there exists $U_2 \in e^*O(X, x)$ such that $f[U_2] \subseteq cl(V)$. By the other hand, X is regular and, therefore, $O(X) = \delta O(X) \subseteq aO(X)$. Now, taking into consideration the Lemma 2.3 and defining $U := U_1 \cap U_2 \in e^*O(X, x)$, we get that $g[U] \subseteq cl(W)$. □

4 Some fundamental properties

Lemma 4.1. *If $f : X \to Y$ is w.e*.c. and $g : Y \to Z$ is continuous, then the composition $g \circ f : X \to Z$ is w.e*.c.*

Proof. Let $x \in X$ and $W \in O(Z, g \circ f(x))$. Since g is continuous, then $g^{-1}[W] \in O(Y, f(x))$. Now, since f is $w.e^*.c.$, we get that there exists $U \in e^*O(X, x)$ such that

$$(g \circ f)[U] \subseteq g[cl(g^{-1}[W])] \subseteq cl(W).$$

□

Lemma 4.2. *Let $f : X \to Y$ be an open δ-continuous surjection and $g : Y \to Z$ a function. If $g \circ f : X \to Z$ is w.e*.c., then g is w.e*.c.*

Proof. Let $V \in O(Z)$. Since $g \circ f$ is $w.e^*.c.$, and taking into consideration the item *1.* from Theorem 3.2, we get that

$$(g \circ f)^{-1}[V] \subseteq cl(int(cl_\delta((g \circ f)^{-1}[cl(V)]))) = cl(int(cl_\delta(f^{-1}[g^{-1}[cl(V)]])))$$

Since f is δ-continuous, and taking account the Lemma 2.5, we obtain that

$$(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \subseteq cl(int(f^{-1}[cl_\delta(g^{-1}[cl(V)])]))$$

Now, since f is surjection, we get that

$$\begin{aligned}
g^{-1}[V] &\subseteq f[cl(int(f^{-1}[cl_\delta(g^{-1}[cl(V)])]))] \\
&\overset{\text{Lemma 2.2(1)}}{=} f[cl_\delta(int(f^{-1}[cl_\delta(g^{-1}[cl(V)])]))] \\
&\overset{f\text{ is }\delta\text{-con.}}{\subseteq} cl_\delta(f[int(f^{-1}[cl_\delta(g^{-1}[cl(V)])])]) \\
&\overset{f\text{ is open}}{\subseteq} cl_\delta(int(f[f^{-1}[cl_\delta(g^{-1}[cl(V)])]])) \\
&\overset{f\text{ is surjection}}{=} cl_\delta(int(cl_\delta(g^{-1}[cl(V)]))) \\
&\overset{\text{Lemma 2.2(1)}}{=} cl(int(cl_\delta(g^{-1}[cl(V)]))).
\end{aligned}$$

Then g is $w.e^*.c.$ by item *1.* from Theorem 3.2. □

Let $\{X_\alpha : \alpha \in I\}$ and $\{Y_\alpha : \alpha \in I\}$ be any two families of topological spaces with the same index set I. The product space of $\{X_\alpha : \alpha \in I\}$ (resp. $\{Y_\alpha : \alpha \in I\}$) is simply denoted by $\prod_{\alpha \in I} X_\alpha$ (resp. $\prod_{\alpha \in I} Y_\alpha$). Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in I$. Let $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be the product function defined as follows: $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \prod_{\alpha \in I} X_\alpha$. The natural projection of $\prod_{\alpha \in I} X_\alpha$ (resp. $\prod_{\alpha \in I} Y_\alpha$) onto X_β (resp. Y_β) is denoted by $p_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ (resp. $q_\beta : \prod_{\alpha \in I} Y_\alpha \to Y_\beta$).

Lemma 4.3. *Let A_α be a subset of X_α for each $\alpha \in I$ and $A = \prod_{i=1}^{n} A_{\alpha_i} \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha$ a nonempty subset of $\prod_{\alpha \in I} X_\alpha$, where n is a positive integer. Then $A \in e^*O\left(\prod_{\alpha \in I} X_\alpha\right)$ if and only if $A_{\alpha_i} \in e^*O(X_{\alpha_i})$ for each $i = 1, 2, \ldots, n$.*

Proof. Let $\alpha \in I$ and $A = \prod_{i=1}^{n} A_{\alpha_i} \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha$, then

$$\begin{aligned}
cl(int(cl_\delta(A))) &= cl\left(int\left(cl_\delta\left(\prod_{i=1}^{n} A_{\alpha_i} \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right)\right)\right) \\
&\overset{\text{Lemma 2.4}}{=} cl\left(int\left[cl_\delta\left(\prod_{i=1}^{n} A_{\alpha_i}\right) \times cl_\delta\left(\prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right)\right]\right) \\
&= cl\left(int\left[\prod_{i=1}^{n} cl_\delta\left(A_{\alpha_i}\right) \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right]\right) \\
&= cl\left[int\left(\prod_{i=1}^{n} cl_\delta\left(A_{\alpha_i}\right)\right) \times int\left(\prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right)\right] \\
&= cl\left[\prod_{i=1}^{n} int\left(cl_\delta\left(A_{\alpha_i}\right)\right) \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right] \\
&= cl\left(\prod_{i=1}^{n} int\left(cl_\delta\left(A_{\alpha_i}\right)\right)\right) \times cl\left(\prod_{\alpha \neq \alpha_i}^{n} X_\alpha\right) \\
&= \prod_{i=1}^{n} cl\left(int\left(cl_\delta\left(A_{\alpha_i}\right)\right)\right) \times \prod_{\alpha \neq \alpha_i}^{n} X_\alpha
\end{aligned}$$

□

Theorem 4.1. *If $f_\alpha : X_\alpha \to Y_\alpha$ is w.e*.c. for each $\alpha \in I$, then $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ is w.e*.c.*

Proof. Let $x = \{x_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$ and $W \in O(\prod_{\alpha \in I} Y_\alpha, f(x))$, then there exists $J = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq I$. Defining

$$V_\alpha := \begin{cases} V_{\alpha_j} \in O(Y_{\alpha_j}) & , \quad \alpha \in J \\ Y_\alpha & , \quad \alpha \notin J \end{cases}$$

we get that $\prod_{\alpha \in I} V_\alpha \in O(\prod_{\alpha \in I} Y_\alpha, f(x))$ and $\prod_{\alpha \in I} V_\alpha \subseteq W$. Since, for all $\alpha \in I$, f_α is $w.e^*.c.$, then there exists $U_\alpha \in e^*O(X_\alpha, x_\alpha)$ such that $f_\alpha[U_\alpha] \subseteq cl(V_\alpha)$. Now, defining $U := \prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\alpha \notin J} X_\alpha$ and taking into consideration the Lemma 4.3, we get that $U \in e^*O(\prod_{\alpha \in I} X_\alpha, x)$ and, therefore

$$f[U] \subseteq \prod_{j=1}^{n} f_\alpha\left[U_{\alpha_j}\right] \times \prod_{\alpha \notin J} Y_\alpha \subseteq \prod_{j=1}^{n} cl\left(V_{\alpha_j}\right) \times \prod_{\alpha \notin J} Y_\alpha \subseteq cl(W).$$

□

Corollary 4.1. *If in addition* $\prod_{\alpha \in I} X_\alpha$ *is regular, then the converse of Theorem 4.1 is true.*

Proof. Let f be $w.e^*.c.$ and $\beta \in I$. Since q_β is continuous, and taking into consideration the Lemma 4.1, we get that $q_\beta \circ f$ is $w.e^*.c.$ Besides, $f_\beta \circ p_\beta = q_\beta \circ f$ and therefore

$$f_\beta \circ p_\beta \text{ is } w.e^*.c. \tag{3}$$

By the other hand $\prod_{\alpha \in I} X_\alpha$ is regular, then

$$p_\beta \text{ is continuous} \Leftrightarrow p_\beta \text{ is } \delta\text{-continuous} \tag{4}$$

So, by equations 3 and 4 and taking into consideration the Lemma 4.2, we get that f_β is $w.e^*.c.$ □

Theorem 4.2. *If* $f : X \to Y$ *is* $w.e^*.c.$ *and* $g : X \to Y$ *is* $w.a.c.$ *and* Y *is Urysohn, then the set* $A = \{x \in X : f(x) = g(x)\}$ *is* e^**-closed in* X*.*

Proof. Let $x \notin A$, then $f(x) \neq g(x)$. As Y is Urysohn, then there exists $V \in O(Y, f(x))$ and $W \in O(Y, g(x))$ such that $cl(V) \cap cl(W) = \emptyset$. Since f is $w.e^*.c.$ and g is $w.a.c.$, then there exists $G \in e^*O(X, x)$ and $H \in aO(X, x)$ such that $f[G] \cap g[H] \subseteq cl(V) \cap cl(W) = \emptyset$. Now, defining $U := G \cap H$ and taking into consideration the Lemma 2.3, we get that $U \in e^*O(X, x)$ and $f[U] \cap g[U] \subseteq f[G] \cap g[H] = \emptyset$. So, $U \in e^*O(X, x)$ and $U \cap A = \emptyset$. Therefore $x \notin e^*\text{-}cl(A)$. □

Theorem 4.3. *If* $f : X \to Y$ *is a* $w.e^*.c.$ *surjection and* X *is* e^**-connected, then* Y *is connected.*

Proof. Suppose that Y is not connected, then there exists $V, W \in \tau \setminus \{\emptyset\}$ such that $V \cap W = \emptyset$ and $V \cup W = Y$, therefore $V, W \in CO(X) \setminus \{\emptyset\}$. Since f is $w.e^*.c.$, and taking into consideration the item *7.* from Theorem 3.2, we get that

$$f^{-1}[V] \subseteq e^*\text{-}int(f^{-1}[cl(V)]) = e^*\text{-}int(f^{-1}[V])$$

$$f^{-1}[W] \subseteq e^*\text{-}int(f^{-1}[cl(W)]) = e^*\text{-}int(f^{-1}[W])$$

Also, $f^{-1}[V \cap W] = \emptyset$ and $f^{-1}[V \cup W] = X$. But f is surjection and, therefore, we get that $f^{-1}[V], f^{-1}[W] \in e^*O(X) \setminus \{\emptyset\}$ with

$$f^{-1}\left[V\right] \cap f^{-1}[W\right] = \emptyset \quad \text{and} \quad f^{-1}\left[V\right] \cup f^{-1}[W\right] = X.$$

□

Corollary 4.2. *If* $f : X \to Y$ *is an* $a.e^*.c.$ *surjection and* X *is* e^**-connected, then* Y *is connected.*

Proof. It is obvious from the fact that almost e^*-continuity implies weakly e^*-continuity. □

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